

SHORT COMMUNICATION

On the Fundamental Solution of the Reduced Wave Equation

M.A. Al-Gwaiz

*Department of Mathematics, College of Science, King Saud University,
P.O. Box 2455, Riyadh 11451, Saudi Arabia*

(Received 12/2/1994; Accepted for publication 27/4/1994)

Abstract. A method is presented for constructing fundamental solutions for the Helmholtz operators $\Delta \pm k^2$ in \mathbb{R}^n in terms of the fundamental singularity for the Laplacian Δ . The feasibility of representing a fundamental solution for $\Delta^2 - k^4$ by forming convolutions of such solutions is also discussed.

Introduction

Next to the Laplacian operator $\Delta = \sum_{k=1}^n \partial^2 / \partial x_k^2 = \sum_{k=1}^n \partial_k^2$ in \mathbb{R}^n , the Helmholtz operator $\Delta + \lambda$ where λ is a real constant, plays a central role in the formulation of physical laws, most notably those involving wave phenomena. The differential operator $\Delta + \lambda$ comes up most naturally when the wave equation $\Delta u + u_{tt} = f$ is reduced to the non-homogeneous Helmholtz equation

$$(\Delta + \lambda)u = f \tag{1}$$

under the assumption that u satisfies $u_{tt} = \lambda u$. Hence (1) is also known as the reduced wave equation. The case when $\lambda = 0$ yields Poisson's equation whose solution is the potential function generated by f , and will not concern us in this investigation. Otherwise λ may be expressed as $\lambda = \pm k^2$ for some positive number k .

In \mathbb{R}^2 equation (1) describes the small vibrations of a stretched elastic membrane, which is subjected to a lateral force whose density per unit area is determined by f .

The corresponding vibrations of a thin elastic plate due to pure bending are governed, in the linear theory of elasticity (see [1] for example), by the differential equation $\Delta^2 + (\rho/D)u_{tt} = f/D$ which reduces to

$$(\Delta^2 - k^4) u = g, \quad (2)$$

where $k^4 = \rho\omega^2/D$. Here ρ is the mass density per unit area of the plate, D its flexural rigidity, and ω is the vibration frequency. Since we can write $\Delta^2 - k^4 = (\Delta - k^2)(\Delta + k^2)$, we are led to consider the pair of equations

$$(\Delta + k^2)u = v, (\Delta - k^2)v = g, \quad (3)$$

both being of the type (1)

We shall consider equation (1) to be valid, in the sense of distributions, in an open connected set Ω in \mathbf{R}^n (see [2] for example). This allows us to admit such highly discontinuous forces as concentrated impulses. If f is a distribution of compact support in the space of distributions $D'(\Omega)$ then the general solution of (1) is $u_0 + F_n * f$. Here u_0 is the general solution of the corresponding homogeneous equation $(\Delta + \lambda)u = 0$ and F_n is a fundamental solution of the Helmholtz operator $\Delta + \lambda$ in \mathbf{R}^n , in the sense that

$$(\Delta + \lambda)F_n = \delta, \quad (4)$$

δ being the Dirac measure in \mathbf{R}^n supported at the point 0. When $F_n f$ is a locally integrable function in \mathbf{R}^n , the convolution product $F_n * f$ is defined by

$$F_n * f(x) = \int_{\text{supp } f} F_n(x - x') f(x') dx'.$$

Otherwise, the reader can refer to [3] or [2] for the distributional definition.

The homogeneous component u_0 of u is determined by appropriate boundary conditions and will not concern us in this treatment. The other component, $F_n * f$, can be constructed once a fundamental solution of $\Delta + \lambda$ is available. Consequently the problem of solving equation (1) is essentially a problem of determining a distribution F_n in \mathbf{R}^n that satisfies (4). This problem can be solved by using the Fourier transformation, a procedure that is both long and tedious (see [3, p.472] or [4, p.49]). The corresponding computation of the fundamental solution for Δ , on the other hand, is straightforward. The solution of $\Delta E_n = \delta$ is given by

$$E_n(x) = \begin{cases} xH(x) & \text{in } \mathbb{R} \\ \frac{1}{2\pi} \log |x| & \text{in } \mathbb{R}^2 \setminus \{0\} \\ -\frac{1}{(n-2)\sigma_n|x|^{n-2}} & \text{in } \mathbb{R}^n \setminus \{0\}, n \geq 3, \end{cases} \tag{5}$$

where $H(x)$ is the Heaviside function and σ_n is the surface area of the unit sphere in \mathbb{R}^n (see [2], p. 151).

In view of the fact that Δ is the principal component of $\Delta + \lambda$, the singular part of $F_n(x)$ should behave like $E_n(x)$ as $x \rightarrow 0$, the point $x = 0$ being an energy sink of the same magnitude for both “potentials”. It is therefore reasonable to expect that F_n can be expressed in terms of E_n . The main purpose in this study is to show how this can be done. With the fundamental solution for the Helmholtz operator at our disposal we can also obtain a solution for the equation (2) of the vibrating plate. For if F_n^+ and F_n^- are fundamental solutions for $\Delta + k^2$ and $\Delta - k^2$, respectively, then $F_n^+ * F_n^-$, provided this convolution exists, is a fundamental solution for $\Delta^2 - k^4$. This follows from the observation that

$$\begin{aligned} (\Delta^2 - k^4) F_n^+ * F_n^- &= (\Delta + k^2) F_n^+ * (\Delta - k^2) F_n^- \\ &= \delta * \delta \\ &= \delta. \end{aligned}$$

Consequently the convolution $(F_n^+ * F_n^-) * g$, where g has compact support, satisfies equation (2).

The Special Cases of $n = 1, 2$

(i) $n = 1$

In this case equation (4) takes the form

$$F_1'' + \lambda F_1 = \delta. \tag{6}$$

Assuming that $F_1(x) = A(x)E_1(x) + B(x)$, where both A and B are analytic functions in \mathbb{R} and $E_1(x) = xH(x)$, equation (6) implies

$$(A'' + \lambda A)xH + 2A'H + A(0)\delta + B'' + \lambda B = \delta,$$

from which we can conclude that $B \equiv 0$, $A(0) = 1$, and $(xA)'' + \lambda xA = 0$. Hence $A(x) = \sin \sqrt{\lambda x} / \sqrt{\lambda x}$ when $x \neq 0$, and $F_1(x) = 1 / \sqrt{\lambda} H(x) \sin \sqrt{\lambda x}$ for all $x \in \mathbb{R}$. Thus

$$F_1(x) = \begin{cases} \frac{1}{k} H(x) \sin kx = F_1^+(x) & \text{if } \lambda = k^2 > 0 \\ \frac{1}{k} H(x) \sinh kx = F_1^-(x) & \text{if } \lambda = -k^2 < 0. \end{cases} \tag{7}$$

Since F_1^+ and F_1^- both have their supports in $[0, \infty)$, the convolution product $F_1^+ * F_1^-$ is a well defined distribution in $D'(\mathbb{R})$ (see [2]) which represents a fundamental solution for the operator $\frac{d^4}{dx^4} - (k^4)$. It is in fact given by the continuous function $H(x) (\sinh kx - \sin kx) / 2k^3$, which is a C^∞ function in $\mathbb{R} \setminus \{0\}$.

It is clear from the above representation of F_1 that we retrieve the fundamental singularity of the Laplacian if we let $\lambda \rightarrow 0$. We also note that F_1^- cannot be obtained from equation (6) by a direct application of the Fourier transformation, since F_1^- is not a tempered distribution.

(ii) $n = 2$

Once again, let $F_2(x) = A(x)E_2(x) + B(x)$ with A and B analytic in \mathbb{R}^2 . The representation (5) shows that E_n is spherically symmetric function in $\mathbb{R}^n \setminus \{0\}$, in the sense that it depends only on $r = |x|$, and we shall assume that the same is true of A and B . The substitution into equation (4) yields the pair of equations $(\Delta + \lambda)A = 0$ and $A(0) = 1$, which means that $A(r)$ is the Bessel function $J_0(kr)$ or $I_0(kr)$, depending on whether $\lambda = k^2$ or $-k^2$. The function B is then any solution of $(\Delta + \lambda)B + (1/\pi r)A' = 0$. The simplest form of F_2 is given in terms of the Bessel functions Y_0 and K_0 by

$$F_2(r) = \begin{cases} \frac{1}{4} Y_0(kr) = F_2^+(r) & \text{if } \lambda = k^2 \\ -\frac{1}{2\pi} K_0(kr) = F_2^-(r) & \text{if } \lambda = -k^2 \end{cases} \tag{8}$$

Since F_2 is a C^∞ function in $\mathbb{R}^2 \setminus \{0\}$ with a logarithmic singularity at $r = 0$, and since $Y_0(kr)$ is of order $1/\sqrt{kr}$ while $K_0(kr)$ is of order e^{-kr} / \sqrt{kr} as $r \rightarrow \infty$, the convolution

$$\begin{aligned}
 F_2^+ * F_2^- (r, \theta) &= - \frac{1}{8\pi} \int_{R^2} Y_0(k | re^{i\theta} - r'e^{i\theta'} |) K_0(kr') r' dr' d\theta' \\
 &= - \frac{1}{8\pi} \int_0^\infty \int_\theta^{\theta+2\pi} Y_0(k | r - r'e^{i\alpha} |) K_0(kr') r' d\alpha dr' \\
 &= - \frac{1}{8\pi} \int_0^\infty \int_0^{2\pi} Y_0(k | r - r'e^{i\alpha} |) K_0(kr') r' d\alpha dr' \tag{9}
 \end{aligned}$$

is a continuous function in R^2 , which is independent of θ and of class C^∞ outside the origin. As a distribution it satisfies $(\Delta^2 - k^4) (F_2^+ * F_2^-) = \delta$ and is therefore a fundamental solution for the operator $\Delta^2 - k^4$.

The General Case of $n \geq 3$

With $x \in R^n$ we shall use (x, x^{n+1}) to denote the corresponding point in R^{n+1} . For the sake of clarity we shall write Δ_n and Δ_{n+1} to denote the Laplacian operator in R^n and R^{n+1} respectively. If δ_n is the Dirac distribution in R^n and h is a distribution in R then the tensor product $h \otimes \delta_n$ is a well defined distribution in R^{n+1} (see [2]).

Taking $\lambda = -k^2$ in equation (4), we have $(\Delta_n - k^2) F_n^- = \delta_n$. On the other hand

$$\begin{aligned}
 \Delta_{n+1} e^{ikx_{n+1}} F_n^- (x) &= (\Delta_n + \partial_{n+1}^2) e^{ikx_{n+1}} F_n^- (x) \\
 &= e^{ikx_{n+1}} (\Delta_n - k^2) F_n^- (x) \\
 &= e^{ikx_{n+1}} \otimes \delta_n
 \end{aligned}$$

Consequently

$$\begin{aligned}
 e^{ikx_{n+1}} F_n^- (x) &= E_{n+1} * (e^{ikx_{n+1}} \otimes \delta_n) \\
 &= E_{n+1} * e^{ikx_{n+1}},
 \end{aligned}$$

provided this last convolution, which is taken with respect to x_{n+1} , exists. But this follows from the fact that $e^{ikx_{n+1}}$ is bounded while $F_{n+1} (x, x_{n+1})$ is integrable with respect to x_{n+1} in R provided $x \neq 0$. Thus

$$\begin{aligned}
 F_n^-(x) &= e^{-ikx_{n+1}} \int_{-\infty}^{\infty} e^{ik(x_{n+1}-x'_{n+1})} E_{n+1}(x, x'_{n+1}) dx'_{n+1} \\
 &= \int_{-\infty}^{\infty} e^{-ikx'_{n+1}} E_{n+1}(x, x'_{n+1}) dx'_{n+1},
 \end{aligned}
 \tag{10}$$

which is the Fourier transform of $E_{n+1}(x, x_{n+1})$ with respect to the variable x_{n+1} , evaluated at (x, k) .

To check the validity of the representation (10) when $n = 3$, recall that $E_4(x, x_4) = -1/4\pi^2(|x|^2 + x_4^2)$, and therefore the right-hand side of (10) is

$$-\frac{1}{4\pi^2} \int_{-\infty}^{\infty} e^{-ikx'_4} \frac{1}{x_4^2 + |x|^2} dx'_4$$

This integral can be evaluated by residues to give $-\frac{1}{4\pi|x|} e^{-k|x|}$. Noting that $-\frac{1}{4\pi|x|} = E_3(x)$, it is simple exercise to verify that $(\Delta_3 - k^2) e^{-k|x|} E_3(x) = \delta_3$.

In general, when $n \geq 3$, we can use the representation (5)

$$E_{n+1}(x, x'_{n+1}) = - \frac{1}{(n-1)\sigma_{n+1}(x'^2_{n+1} + |x|^2)^{\frac{n-1}{2}}}$$

and the Fourier transform formula (see [5]).

$$\int_{-\infty}^{\infty} e^{-ikt} \frac{1}{(t^2 + |x|^2)^v} dt = \frac{2\sqrt{\pi}}{\Gamma(v)} \left(\frac{k}{2|x|} \right)^{v-\frac{1}{2}} K_{v-\frac{1}{2}}(k|x|),$$

which is valid for all $k > 0$ and $v > 0$, to arrive at

$$F_n^-(x) = - \frac{1}{(n-1)\sigma_{n+1}} \frac{2\sqrt{\pi}}{\Gamma\left(\frac{n-1}{2}\right)} \left(\frac{k}{2|x|} \right)^{\frac{n-1}{2}-1} K_{\frac{n-1}{2}}(k|x|), \quad n \geq 3, \tag{11}$$

which is a C^∞ function in $\mathbb{R}^n \setminus \{0\}$. A comparison with the result of the previous section shows that the representation (11) is also valid when $n = 2$. As $|x| \rightarrow 0$, we know that $K_{\frac{n}{2}+1}(k|x|)$ behaves like $2^{\frac{n}{2}-2} \Gamma\left(\frac{n}{2}-1\right) / |x|^{\frac{n}{2}-1}$, hence $F_n^-(x)$ has a singularity at $x = 0$ order $1/|x|^{n-2}$, which is of the same order as $E_n(x)$, as would be expected. On the other hand, as $|x| \rightarrow \infty$, $K_{\frac{n}{2}-1}(k|x|)$ behaves like $\sqrt{\pi} e^{-k|x|} / \sqrt{2k|x|}$ so $F_n^-(x)$ decays like $e^{-k|x|} / |x|$.

Formally, $F_n^+(x)$ is obtained from $F_n^-(x)$ by replacing k by $\pm ik$, but this produces a divergent integral in (10), so the representation (10) cannot be used to compute $F_n^+(x)$. In (11), however the transformation $k \rightarrow ik$ leads to the expression

$$F_n^+(x) = - \frac{1}{(n-1)\sigma_{n+1}} \frac{2\sqrt{\pi}}{\Gamma\left(\frac{n-1}{2}\right)} \left(\frac{ik}{2|x|}\right)^{\frac{n}{2}-1} K_{\frac{n}{2}-1}(ik|x|),$$

which is a well-defined C^∞ function in $\mathbb{R}^n \setminus \{0\}$. As a distribution F_n^+ satisfies $(\Delta_n + k^2) F_n^+ = \delta_n$. It has the appropriate singularity at $x = 0$, namely of the same order as E_n , and behaves like $e^{-ik|x|} / |x|^{\frac{n-1}{2}}$ as $|x| \rightarrow \infty$. In \mathbb{R}^3 , it gives the expected

$$\text{result } F_3^+(x) = - \frac{1}{4\pi|x|} e^{-ik|x|} = E_3(x) e^{-ik|x|}$$

Fundamental Solution for $\Delta_n^2 - k^4$

Based on the asymptotic behaviour of F_n^- and F_n^+ near $x = 0$ and $|x| = \infty$, the convolution product $F_n^+ * F_n^-(x)$ is a well defined C^∞ function in $\mathbb{R}^n \setminus \{0\}$ which constitutes a fundamental solution for $\Delta_n^2 - k^4$. In \mathbb{R}^3 the product $F_3^+ F_3^-(x)$ has a singularity of order $|x|^{-2}$ at $x = 0$, and is therefore locally integrable. Hence $F_3^+ * F_3^-$ is continuous in \mathbb{R}^3 . The (uniform) convergence of the convolution integral which represents $F_n^+ * F_n^-(x)$ in $\mathbb{R}^2 \setminus \{0\}$ is forced by the exponential decay of F_n^- , the fundamental solution for $\Delta_n - k^2$, due to the factor $e^{-k|x|}$. When $k = 0$ and $F_n^- = F_n^+ = E_n$, $E_n * E_n$ is, in general, not defined. But the fundamental singularity for Δ_n^2 , which we shall denote by S_n , can nevertheless be obtained by solving $\Delta_n S_n = E_n$. Assuming S_n is spherically symmetric, this is the simple ordinary differential equation $(r^{n-1} S_n')' = E_n$ whose singular solutions in $\mathbb{R}^n \setminus \{0\}$ are given by

$$\begin{aligned} & \frac{1}{6} x^3 H(x) && \text{if } n = 1 \\ & \frac{1}{8\pi} |x|^2 \log|x| && \text{if } n = 2 \end{aligned}$$

$$-\frac{1}{8\pi} |x| \quad \text{if } n = 3$$

$$-\frac{1}{8\pi^2} \log |x| \quad \text{if } n = 4$$

$$\frac{1}{2(n-2)(n-4)\sigma_n} \cdot \frac{1}{|x|^{n-4}} \quad \text{if } n \geq 5.$$

S_n being the singular part of $F_n^+ * F_n^-$. This is consistent with the observation that the convolution integral representing $F_n^+ * F_n^-$ is continuous when $n = 1, 2$ and 3 .

References

- [1] Timoshenko, S. *Theory of Plates and Shells*. New York: McGraw-Hill, 1959.
- [2] Al-Gwaiz, M.A. *Theory of Distributions*. New York: Marcel Dekker, 1992.
- [3] Szmydt, S. *Fourier Transformation and Linear Differential Equations*. Dordrecht-Holland: Reidel, 1977.
- [4] Treves, F. *Basic Linear Partial Differential Equations*. New York: Academic Press, 1975.
- [5] Magnus, W.; Oberhettinger, F. and Soni, R.P. *Formulas and Theorems for the Special Functions of Mathematical Physics, 3rd ed.* Berlin: Springer-Verlag, 1966.

حول الحل الأساسي لمعادلة التموج

محمد عبدالرحمن القويز

قسم الرياضيات، كلية العلوم، جامعة الملك سعود، ص.ب. ٢٤٥٥،

الرياض ١١٤٥١، المملكة العربية السعودية

(استلم في ١٢/٢/١٩٩٤ م؛ قبل للنشر في ٢٧/٥/١٩٩٤ م)

ملخص البحث. يعرض هذا البحث طريقة للحصول على الحلول الأساسية لمؤثر هلمهولتز في الفضاء الاقليدي متعدّد الأبعاد، وذلك بالاستناد إلى الحل الأساسي لمؤثر لابلاس المعروف. وانطلاقاً من ذلك يناقش البحث إمكانية تكوين الحلول الأساسية لمعادلة تذبذب الصفائح المرنة.