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The Unsteady Motion of Two Spheres in a Viscous Fluid at Low Reynolds Numbers

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Abstract. Matched asymptotic solutions are obtained, in the Stokes regime, for the time-dependent viscous incompressible flow past two spheres moving along their line of centres at very low Reynolds Numbers. The correction of O(1/d) in the drag force on a sphere due to the presence of the second sphere is estimated.

1. Introduction

The application of hydrodynamic theory to the behaviour of solid and liquid particles moving in a viscous medium at low Reynolds numbers has received increased attention in recent years in connection with problems in chemical, geological, mining and bio-medical engineering. An excellent review summarising the current state of knowledge in this field and its applications is given by Brenner [1].

An exact solution of the linearised Navier-Stokes equations for the steady axisymmetric motion of a viscous fluid at low Reynolds Numbers, corresponding to two spheres translating with equal velocities, has been obtained by Stimson and Jeffery [2] using a bipolar coordinate transformation. The problem was formulated for spheres of arbitrary size but results were presented only for the equal spheres. Abdel Moneim [3] has presented a theory for a system of rigid bodies moving through an unbounded ideal fluid. He obtained hydrodynamic forces acting on the bodies by applying Lagrange's equation. Recently, Rushton an Davies [4] have solved the unsteady problem of two fluid spheres slowly moving along their line of centres.

In most technical applications, multiple particle systems are more important than the single particle situation. The main purpose of this paper is to develop an understanding of the two body system starting with the dynamics of a single particle. In this paper an attempt is made to study the unsteady flow past two equal spheres, placed at a distance 'ad' apart, using the linear unsteady equations as the spheres move from rest along their line of centres. The method of reflections as proposed by Happel and Brenner [5], is used to solve this two-body problem in hydrodynamics. The inertial effects at low Reynolds numbers are accounted for by adopting the method of matched asymptotic technique. The correction of O(1/d) to the drag force due to the presence of the second sphere is estimated.

2. Formulation of the Problem

Consider two equal spheres with radius 'a', separated by a distance 'ad', moving along their line of centres with velocity V(t) in an unbounded viscous, incompressible medium. The fluid at infinity is at rest. The frame of reference is the spherical polar system with origin at the centre of the spheres and the polar axis along the Z-direction (Fig. 1).

The radius, pressure, time and stream function are expressed non-dimensionally by;

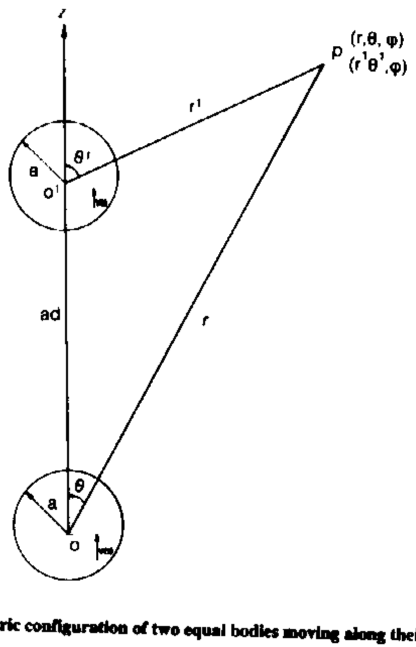

Fig. 1. Geometric configuration of two equal bodies moving along their line of centres

$$r = \frac{r^*}{a}, \; p = \frac{ap^*}{U\varrho\gamma}, \; t = \frac{\gamma t^*}{a^2}, \; \psi = \frac{\psi^*}{Ua^2}, \tag{1}$$

the flow problem is governed by the following non-dimensionalised differential equation as shown by Goldstein [6],

$$\left(\frac{\partial}{\partial t} - D^2\right) D^2 \psi = \frac{Re}{r^2}\left[\frac{\partial(D^2\psi,\psi)}{\partial(r,\mu)} - 2 D^2 \psi L\psi\right], \tag{2}$$

where :

$$D^2 = \frac{\partial^2}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2}{\partial \mu^2}; \; L = \frac{\mu}{1-\mu^2}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \mu} \tag{3}$$

Here the Reynolds number $R_e = Ua/\gamma$ (γ being the kinematic viscosity), $\mu = \cos\theta$, ϱ is the density of fluid, p is the pressure, t is the time variable and ψ is the stream function. The appropriate boundary and initial conditions satisfied by ψ are:

$$\frac{\partial\psi}{\partial\mu} = \frac{\partial\psi}{\partial r} = 0 \quad , \quad \text{at } r = 1, \tag{4}$$

$$\frac{\partial\psi}{\partial\mu'} = \frac{\partial\psi}{\partial r'} = 0 \quad , \quad \text{at } r' = 1, \tag{5}$$

and

$$\psi \approx -\frac{1}{2}V(t)\begin{cases} r^2(1-\mu^2)\,H(t), & \text{as } r \longrightarrow \infty; \\ r'^2(1-\mu'^2)\,H(t), & \text{as } r' \longrightarrow \infty. \end{cases} \tag{6}$$

Here $\mu' = \cos\theta'$ and H(t) is the Heaviside step function.

3. Solutions

Assuming $R_e << 1$, the Stokes approximation to equation (2)

$$\left(\frac{\partial}{\partial t} - D^2\right) D^2\psi = 0 \tag{7}$$

Taking Laplace transform with respect to t of equation (7), one gets

$$(s - D^2)\, D^2 \bar{\psi} = 0 \quad , \tag{8}$$

where $\bar{\psi}$ (r,θ,s) denotes the Laplace transform of ψ(r,θ,t). A general solution of (8) can be written as [7]

$$\bar{\psi} = \sum_{n=1}^{\infty} [\bar{A}_n(s) r^{n+1} + \bar{B}_n(s) r^{-n} + \bar{C}_n(s) r^{1/2} I_{(n+1)/2}(s^{1/2} r) + \bar{D}_n(s) r^{1/2} K_{(n+1)/2}(s^{1/2} r)] P_1^1(\mu) P_n^1(\mu) \tag{9}$$

Where $I_{(n+1)/2}$ and $K_{(n+1)/2}$ are modified Bessel functions of first and second kind respectively and $P_n(\mu)$ is the polynomial, $P_n^1(\mu)$ is the associated Legrendre function. Introducing the conditions of equation (6), the solution takes the form

$$\psi = \begin{cases} -(1/2)\bar{V}(s) r^2 (1-\mu^2 \\ -(1/2)\bar{V}(s) r'^2 (1-\mu'^2) \end{cases} + \sum_{n=1}^{\infty} [\bar{B}_n(s) r^{-n} + \bar{D}_n(s) r^{1/2} K_{(n+1)/2}(s^{1/2} r)] P_1^1(\mu) P_n^1(\mu)$$
$$+ \sum_{m=1}^{\infty} [\bar{M}_m(s) r'^{-m} + \bar{N}_m(s) r'^{1/2} K_{(m+1)/2}(s)^{1/2} r'] P_1^1(\mu') P_m^1(\mu') \tag{10}$$

As shown by Whittakar and Watson [8] and Watson [9], following identities are valied for r, r′ < d :

$$\frac{P_n^1(\mu')}{r'^{n+1}} = \frac{(-1)^{n+1}}{d^{n+1}} \sum_{j=1}^{\infty} \frac{(n)_{j+1}}{(j+1)!} \left(\frac{r}{d}\right)^j P_j^1(\mu); \tag{11a}$$

$$\frac{P_n^1(\mu)}{r^{n+1}} = \frac{1}{d^{n+1}} \sum_{j=1}^{\infty} (-1)^{j+1} \frac{(n)_{j+1}}{(j+1)!} \left(\frac{r}{d}\right)^j P_j^1(\mu'); \tag{11b}$$

$$\frac{K_{(m+1)/2}(xr')}{\sqrt{xr'}} P_m^1(\mu') = \sum_{n=1}^{\infty} X_n^m(xd) \frac{I_{(n+1)/2}(xr)}{\sqrt{xr}} P_n^1(\mu); \tag{12a}$$

$$\frac{K_{(m+1)/2}(xr)}{\sqrt{xr}} P_m^1(\mu) = \sum_{n=1}^{\infty} Y_n^m(xd) \frac{I_{(n+1)/2}(xr')}{\sqrt{xr'}} P_n^1(\mu'); \tag{12b}$$

where : $(n)_s = n(n+1)(n+2) \ldots\ldots (n+s-1)$; (13a)

$$A_m^o = \sqrt{\frac{\pi}{2}}\,(2m+1)\,\frac{K_{(m+1)/2}(xd)}{\sqrt{xd}}\,; \tag{13b}$$

$$A_m^1 = \sqrt{\frac{\pi}{2}}\,(2m+1)\,[m\,\frac{K_{(m+1)/2}(xd)}{(xd)^{3/2}} - \frac{K_{(m+3)/2}(xd)}{\sqrt{xd}}\,]; \tag{13c}$$

$$A_m^n = \frac{2n-1}{n}\,\frac{\partial}{\partial(xd)}\,A_m^{n-1} - \frac{n-1}{n}\,A_m^{n-2}\,, \quad \text{for } n > 2\,; \tag{13d}$$

$$X_n^m = A_n^m + \frac{xd}{2n-1}\,A_{n-1}^m - \frac{xd}{2n+3}\,A_{n+1}^m\,; \tag{13e}$$

$$Y_n^m\,(xd) = (-1)^{n+m}\,X^m{}_n\,(xd)\,. \tag{13f}$$

In the light of relations (11a, b) and (12a, b), the solution in equation (10) can be written as

$$\begin{aligned}\bar{\psi} = &-(1/2)\,\bar{V}(s)\,r^2\,(1-\mu^2) + \sum_{n=1}^{\infty}\,[\bar{B}_n(s)\,r^{-n} + \bar{D}_n(s) \\ &\qquad \times r^{1/2}\,K_{(n+1)/2}\,(s^{1/2}r)]\,P_1^1(\mu)\,P^1{}_n(\mu) \\ &+ \sum_{m=1}^{\infty}\,\sum_{n=1}^{\infty}\,(-1)^{m+1}\,\frac{(m)_{n+1}}{(n+1)!}\,\frac{r^{n+1}}{d^{m+n+1}}\,P_1^1(\mu)\,P_n^1(\mu) \\ &+ \sum_{m=1}^{\infty}\,\sum_{n=1}^{\infty}\,\bar{N}_m(s)\,X_n^m(s^{1/2}d)\,I_{(n+1)/2}(s^{1/2}r)r^{1/2}\,P_1^1(\mu)\,P_n^1(\mu)\end{aligned} \tag{14a}$$

Similarly for the second sphere ,

$$\begin{aligned}\bar{\psi} = &-(1/2)\,\bar{V}(s)\,r'^2\,(1-\mu'^2) \\ &+ \sum_{n=1}^{\infty}\,\sum_{m=1}^{\infty}\,(-1)^{m+1}\,\bar{B}_n(s)\,\frac{(n)_{m+1}}{(m+1)!}\,\frac{r'^{m+1}}{d^{m+n+1}}\,P_1^1(\mu')\,P_m^1(\mu') \\ &+ \sum_{n=1}^{\infty}\,\sum_{m=1}^{\infty}\,\bar{D}_n(s)\,Y^n{}_m\,(s^{1/2}d)r'^{1/2}\,I_{(m+1)/2}\,(s^{1/2}r')\,P_1^1(\mu')\,P_m^1(\mu')\end{aligned}$$

$$+ \sum_{m=1}^{\infty} [\bar{M}_m(s)\, r'^{-m} + \bar{N}_m(s) r'^{1/2} K_{(m+1)/2}(s^{1/2} r')]\, P_1^1(\mu')\, P_m^1(\mu') \tag{14b}$$

Use of boundary conditions in equations (4) and (5), the solutions in equations (14a,b) give rise to eight compatiable simultaneous equations. Neglecting the terms of $O(1/d^n)$, $n > 4$, for simplicity, the constants $\bar{B}_n$, $\bar{D}_n$, $\bar{M}_m$, $\bar{N}_m$ can be calculated.

The drag on the sphere is given by [7],

$$f(t) = 2\pi\, \varrho U a \gamma \int_{-1}^{+1} \Big[-p\mu - 2\mu \frac{\partial}{\partial r}\Big(\frac{1}{r^2}\frac{\partial \psi}{\partial \mu}\Big)$$

$$- \frac{1-\mu^2}{r^3}\frac{\partial^2 \psi}{\partial \mu^2}\, \frac{2}{r^2}\frac{\partial \psi}{\partial r}\, \frac{1}{r}\frac{\partial^2 \psi}{\partial r^2} \Big]_{r=1} r^2 d\mu\,, \tag{15}$$

which in a nondimensional form $F(t) = -f(t)/(6\pi\varrho\gamma U a)$ can be expressed as

$$F(t) = \frac{2}{9}\Big[\dot{V}(t) + \dot{B}_1(t) - 2\int_0^t \dot{D}_1(x)\, K(t-x)\, dx$$

$$- \frac{2}{d^3}\dot{M}_1(t) - \int_0^t \sum_{m=1}^{3} \dot{N}(x)\, I(m, t-x)\, dx\Big]\,, \tag{16}$$

where:

$$K(t) = L^{-1}[K_{3/2}(s^{1/2})]\,; \tag{17a}$$

$$I(m,t) = L^{-1}[Y_1^m(s^{1/2} d)\, I_{3/2}(s^{1/2})]\,. \tag{17b}$$

To avoid cumbersome calculations and the difficulties and the difficulties involved in simplifying the summation, the terms of $O(1/d)$ are retained in the calculation of the drag, *i.e.*

$$F(t) = (1/3)\Big[V(t) + \dot{V}(t) + \int_0^t \frac{3\dot{V}(x)}{\sqrt{\pi(t-x)}}\, dx\Big]$$

$$- (1/2)d\Big[3V(t) + \int_0^t \frac{3V(x)}{\sqrt{\pi(t-x)}}\,(2\, e^{-1/(t-x)} - 1)\, dx\Big] \tag{18}$$

4. Inertial Effects

Following Sano [10] the time and space region of the flow is subdivided into three domains.

In the small-time domain where $t = O(1)$ the vorticity layer is confined to the inner region in r and the far field remains relatively at rest. Therefore, the assumption that the nonlinear inertia terms are negligible is valid throughout the flow field and the corresponding expansion of the stream function can be written as

$$\psi = \psi_0(r,\mu,t) + R_e\psi(r,\mu,t) + \ldots \tag{19}$$

The leading term ψ_0, satisfying the unsteady Stokes equation (7) and the boundary conditions (4-6), is given by equations (14a,b). In the large-time inner domain, where $t = O(R_e^{-2})$ and $r = O(1)$, the time derivative terms along with the nonlinear terms are neglected from the momentum equations since the corresponding motion is quasi-steady. As time increases the vorticity diffuses into the outer domain in r. Consequently in the large-time outer domain, the terms of the momentum equation acquire comparable magnitudes and the large-time inner solution is no more valid in this domain. The two expansions in the large-time domain are constructed in such a way that: 1) the inner expansion satisfies the boundary condition on the surface; 2) the outer expansion satisfies the boundary condition at infinity; 3) the two expansions match identically in the overlapping region in space and also match the small-time expansion (19) at small T, where T is the time variable in the large-time domain defined as:

$$T = R_e^2\, t \tag{20}$$

The governing equations at large-time can be obtained from (20) when $\partial/\partial t$ is replaced by $R_e^2\frac{\partial}{\partial T}$. The appropriate expansion for the inner solution for large-time is

$$\psi^{(i)} = \psi_0{}^{(i)}(r,\mu,T) + R_e\,\psi_1{}^{(i)}(r,\mu\;,T) + O(R_e^2) \tag{21}$$

where $\psi^{(i)}(r,\mu,T)$ is the same as $\psi(r,\mu,T)$ of (19). In the large-time outer domain, the variables are

$$R = R_e r,\ \psi^{(0)}(R,\mu,T) = R_e^2\psi(r,\mu,t) \tag{22}$$

in terms of which the governing equation is

$$\left(\frac{\partial}{\partial T} - \Delta^2\right) \Delta^2 \psi^{(0)} = \frac{1}{R^2}\left[\frac{\partial(\psi^{(0)}, \Delta^2\psi^{(0)})}{\partial(R,\mu)} + 2\Delta^2 \psi^{(0)} M\psi^{(0)}\right], \tag{23}$$

where: $\Delta^2 = R_e^{-2} D^2$; $M = R_e^{-1} L$ (24)

The expansion for $\psi^{(0)}$ is

$$\psi^{(0)} = -(1/2)V(T/R_e^2)\, R^2 (1-\mu^2) + R_e\psi^{(0)}(R,\mu,T)+O(R_e^2) \tag{25}$$

where for sufficiently small R_e, $V(T/R_e^2)$ can be assumed to take its asymptotic value 1.

Substituting (21), into the corresponding equation valid for large-time domain, one gets:

$$D^4\psi_0^{(i)} = 0 \quad ; \tag{26}$$

$$D^4\psi^{(i)} = r^{-2}\left[\frac{\partial(\psi_0^{(i)}, D^2\psi_0^{(i)})}{\delta(r,\mu)} + 2\, D^2\psi_0^{(i)}\, L\, \psi_0^{(i)}\right] \tag{27}$$

It is not difficult to show that the solution of (26) satisfying the boundary conditions on the body and the matching requirements that the contribution of $\psi_0^{(i)}$ to $\psi^{(0)}$, when expressed in terms of outer variables, should not contain terms of greater order than unity. It can be written as

$$\psi_0^{(i)} = \Big[(1/4)\left(-1 + \frac{3}{2d} - \frac{9}{4d^2}\right)(r^{-1} - 3r + 2r^2)$$

$$+ \frac{1}{16d^3}\left(\frac{79}{10} r^{-1} - \frac{57}{2} r + 23r^2 - \frac{12}{5} r^4\right)\Big](1-\mu^2) \tag{28}$$

The second term $\psi_1^{(0)}$ of (25) satisfies

$$\left(\Delta^2 + \frac{1-\mu^2}{R}\frac{\partial}{\partial\mu} + \mu\frac{\partial}{\partial R} - \frac{\partial}{\partial T}\right) \Delta^2 \psi_1^{(0)} = 0 \tag{29}$$

Taking Laplace transform of (29) with respect ot T, one gets

$$\left(\Delta^2 + \frac{1-\mu^2}{R}\frac{\partial}{\partial\mu} + \mu\frac{\partial}{\partial R} - s\right) \Delta^2 \overline{\psi}_1^{(0)} = 0 \tag{30}$$

A solution of (30) which vanishes at infinity is

$$\Delta^2 \bar{\psi}_1^{(o)} = C(s) \exp\left[-\frac{R}{2}(X+\mu)\right]\left(1+\frac{2}{RX}\right)(1-\mu^2) \tag{31}$$

where $X = (4S+1)^{1/2}$. The constant C(S) can be obtained, by matching the right hand side of (31) with the Laplace transformation of the vorticity associated with the inner-field, as

$$C(s) = -\frac{X}{4s}\left(3+\frac{2}{d}\right) \tag{32}$$

$\psi_1^{(0)}$ is now determined from

$$\Delta^2 \bar{\psi}_1^{(0)} = -\frac{X}{4s}\left(3+\frac{2}{d}\right)\left(1+\frac{2}{RX}\right)\exp\left[-\frac{R}{2}(X+\mu)\right](1-\mu^2) \tag{33}$$

It can be verified the the solution of (33), which is not of greater order than unity and satisfies the matching requirement, is

$$\begin{aligned}
\psi_1^{(0)} = \left(\frac{3}{2}+\frac{1}{d}\right)\Big[& -\mu H(T) + \frac{(1+\mu)}{2}\operatorname{erf}\left(\frac{R}{2\sqrt{T}}+\frac{\sqrt{T}}{2}\right) \\
& \times \left[1-\exp\left(\frac{R}{2}(1-\mu)-\frac{(1-\mu)}{2}\operatorname{erf}\left(\frac{R}{2\sqrt{T}}-\frac{\sqrt{T}}{2}\right)\right]\right. \\
& \times \left[1-\exp\left(-\frac{R}{2}(1+\mu)\right)\right] + e^{-\mu R/2}\left[\sin h\left(\frac{R}{2}\right)+\mu \cos h\left(\frac{R}{2}\right)\right] \\
& -(\pi T)^{-1/2}\exp\left(\frac{-R^2}{4T}-\frac{T}{4}\right)\left[e^{-\mu R/2}-\cos h(R/2)+\mu \sin h\left(\frac{R}{2}\right)\right] \\
& -\frac{(RT)^{-1/2}}{2}R^3\exp(-T/4)\sum_{n=1}^{\infty}(-1)^{n+1}(2n+1)^{-1}\left[P_{n+1}(\mu)\right. \\
& \left. - P_{n-1}(\mu)\right]G_n(R,T)\Big],
\end{aligned} \tag{34}$$

where

$$G_n(R,T) = \int_0^1 \left[\left(\frac{R^2X^2}{4T^2}-\frac{T}{2}-\frac{1}{4}\right)I_{(n+1)/2}(Rx/2)\right.$$

$$\times e^{\;Rx/4T} X^{(n+3)/2} + \left(\frac{R^2}{4T^2X^2} - \frac{T}{2} - \frac{1}{4}\right)$$

$$I_{(n+1)/2}(R/2x)\, e^{-R^2/4TX^2} X^{(n-5)/2} \,]\, dx \tag{35}$$

Substituting for $\psi_0^{(i)}$ from (28), the solution of (27) satisfying the boundary condition and the matching requirement (with outer expansion (34)), can be obtained as $\psi_1^{(i)}(r,\mu,T)=$

$$-\frac{1}{32}\left(3+\frac{2}{d}\right)\left[\left(1+\frac{4}{T^2}\right)\operatorname{erf}(T^{1/2}/2) + \frac{2}{\sqrt{\pi}T}\left(1-\frac{2}{T}\right)\right.$$

$$\times\, e^{-T/4}\,](r^{-1} - 3r + 2r^2)\, P_1^1(\mu)\, P_1^1(\mu)$$

$$-\left[\frac{1}{32}(r^{-2} - r^{-1} - 3r + 2r^2 + 1) - \frac{1}{8d}\left(\frac{51}{8}r^{-2}\right.\right.$$

$$-\frac{9}{2}r^{-1} + \frac{21}{4}r^2 - \frac{9}{4}r + 3\Big) - \frac{1}{64d^2}\Big(-\frac{549}{4}r^{-2} + 81\,r^{-1}$$

$$-117\,r^2 + \frac{81}{2}r - \frac{27}{2}\Big) - \frac{1}{80d^3}\Big(9r^{-4} + 2827\,r^{-2}$$

$$-\frac{1047}{8}r^{-1} + \frac{433}{2}r^2 - \frac{171}{3}r + \frac{7179}{8}\Big)\Big]\, P_1^1(\mu)\, P_2^1(\mu). \tag{36}$$

Making use of the inner solution $\psi^{(i)} = \psi_0^{(i)} + R_e\,\psi_1^{(i)}$ given by (28) and (36), the force F up to O(1/d) for the large time domain, can be obtained as

$$F = \left[\left(1-\frac{3}{2d}\right) + R_2\left\{\frac{1}{8}\left(3+\frac{2}{d}\right)\left[\left(1+\frac{4}{T^2}\right)\operatorname{erf}\left(\frac{\sqrt{T}}{2}\right)\right.\right.\right.$$

$$\left.\left.\left. + \frac{2}{\sqrt{\pi T}}\left(1-\frac{2}{T}\right)e^{-T/4}\right]\right\}\right] \tag{37}$$

A single composite expansion for the fluid force, valid for all values of time can be obtained by replacing the first two terms of O(1) in (37) by the force expression (18) which is valid for small time.

5. Conclusions

From the expression (18) one can obtain the single sphere drag of Bentwish and Miloh [7] when $d \to \infty$. Finally, one can note that the presence of second sphere gives the drag a negative correciton of O(1/d) when both the spheres move in the same direction. In such a case, the loss in drag of the first sphere is caused by the acceleration imported to it in the direction of the fluid motion due to the inertia of the second moving spheres.

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الحركة الغير مستقرة لكرتين في مائع لزج لأرقام رينولد الصغيرة

بابولو مانيكيالا راو

المعمل العلمي والتكنولوجي البحري ، فيزا خانيام ٥٣٠٠٠٦ ، الهند

ملخص البحث. تم الحصول على حلول مقاربة ومثيلة للسريان اللزج الغير منضغط والذي يعتمد على الوقت لحركة كرتين في خط مراكزهن عند أعداد رينولد الصغيرة. وتم تعيين التصحيح اللازم O(1/d) في قوة الجر على الكرة نتيجة لتواجد الكرة الثانية.