

The Effect of Suction or Injection on the Laminar Boundary Layer Development Over a Stretched Surface

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Abstract. The similarity solutions for laminar boundary layer equations over a stretched surface with suction or injection are presented. The surface was assumed to move with a power law velocity profile corresponding to a velocity parameter $-0.93416 \leq m \leq 5.0$ and subject to a dimensionless suction or injection controlling parameter $-1.0 \leq d \leq 4.0$. It was found that the suction of the boundary layer on the stretched surface delays the backflow while injection increases the strength of the reverse (or velocity overshoot) flow for more negative values of m . Non-unique solutions of the governing equations subject to the boundary conditions have been obtained for various values of m . The dimensionless shear stress at the stretched surface increases with increasing d up to a maximum, where there is a balance between the shearing force and the inertia force of injection, then it decreases asymptotically to zero for some decelerated flow ($m < 0$). However, for accelerated flow ($m > 0$) it increases asymptotically to zero.

Nomenclature

- A, = Constant = -0.334287 ;
- B, = Constant = 0.406887 ;
- C, = Constant = -0.171006 ;
- E, = Constant = 0.0240451 ;
- d, = Dimensionless blowing (or suction) parameter;
- f, = Dimensionless stream function;
- m, = Velocity exponent parameter;
- Re_x, = Reynolds number [$= U_0 x^{(m+1)} / \nu$ or $= U_w x / \nu$]
- u, = Velocity component in x-direction;
- U₀, = Constant;
- v, = Velocity component in y-direction;

- $x,$ = Coordinate in direction of surface motion;
 $y,$ = Coordinate in direction normal to surface motion.

Greek symbols

- $\delta,$ = Boundary layer thickness;
 $\eta,$ = Dimensionless similarity variable $[= y \sqrt{(m+1)/2} \sqrt{(U_0 x^{m-1}/\nu)}]$;
 $\mu,$ = Absolute viscosity;
 $\nu,$ = Kinematic viscosity;
 $\tau,$ = Shear stress.

Subscripts

- $w,$ = Condition at the surface;
 $\infty,$ = Condition at ambient medium.

Superscripts

- ' = Differentiation with respect to η

Introduction

The laminar boundary layer flow field developed by a continuously stretched surface in a quiescent fluid has many industrial applications. Such applications are continuously extruded polymer sheets and plastic films from a slit, and in the coating of a moving surface in which this surface is isothermal (Rezaian and Poulikakos [1]). Therefore, in this paper the effect of suction or injection of the boundary layer over a stretched surface is studied. The surface is assumed to move with the more general power law velocity profiles.

Sakiadis [2-4] has reported the flow field analysis for a continuous flat and cylindrical surfaces respectively. In his analysis the stretched surface was assumed to move with uniform velocity ($m = 0, d = 0$) and similarity solutions were developed for the governing equations to map out the boundary layer thickness, displacement and momentum thickness respectively. A class of flow field similarity solutions has been considered by Banks [5] for stretching surface with a power law velocity distribution however, suction or injection was not allowed. He also reported that for this type of similarity solutions there is no evidence for existence of dual solutions of the governing equations subject to the boundary conditions. The flow field $m = 0$ corresponding to a surface moving with a uniform speed has been considered by many

authors. The boundary layer of air generated by a sheet of water has been studied by Taylor [6], and the problem of oscillatory viscous flow was reported by Stuart [7].

The flow and temperature fields of a stretching surface ($d = 0$) have been analyzed by Crane [8] for a linearly moving surface, and by Vleggaar [9] for flat and cylindrical surfaces moving with uniform or linear speed. Furthermore, an stretching surface moving with a power law velocity and temperature profiles has recently been studied for three different temperature boundaries by Ali [10].

The suction or blowing of the boundary layer over a stretched surfaced has been reported by Gupta and Gupta [11] for linear surface velocity with uniform temperature. A series solution of the boundary layer equations subject to suction or injection was reported for a vapor boundary layer at the condensate surface by Ackroyd [12].

In the present study a class of similarity solutions of the boundary layer equations of a stretched surface moving with the more general power law velocity subject to suction or injection at the surface is reported.

The mathematical analysis of the problem is presented in section II followed by the numerical solution procedure in section III. Results and discussion are reported in section IV. Finally, summary and conclusions are given in section V.

Mathematical analysis

The motion of laminar incompressible boundary layer flow with constant fluid properties over a stretching surface with suction or injection is governed by the following continuity and momentum equations.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \quad (2)$$

The usual boundary layer approximation of a slender region (Schlichting [13]) has been used in developing the above governing equations with zero axial pressure gradient. Equations (1) and (2) are assumed to have the following boundary conditions

$$u = U_w = U_0 x^m, \quad v = v_w \neq 0 \quad @ y = 0 \quad (3)$$

$$u \rightarrow 0, \quad @ y \rightarrow \infty \quad (4)$$

in which m is the velocity stretching parameter and U_o is constant. The cartesian (x, y) and the boundary layer representation on a stretching surface are sketched in Fig. 1. In this figure the x -axis is in direction of the moving surface, and the y -axis is perpendicular to it. The velocity component in the direction of x and y are u and v respectively. Qualitative vertical velocity distributions are presented in Fig. 1 for three different values of m which presents an injection flow at the surface boundary. Equations (1) and (2) may be reduced to one single ordinary differential equation by using the following similarity transformation

$$u = U_o x^m f'(\eta), \quad \eta = y \sqrt{\frac{m+1}{2}} \sqrt{\frac{U_o x^m}{\nu x}} \quad (5)$$

$$v = -\sqrt{\frac{2\nu U_o}{m+1}} x^{\frac{m-1}{2}} \left[\frac{m+1}{2} f + \frac{m-1}{2} f' \eta \right] \quad (6)$$

with a prime denoting differentiation with respect to η , η is the similarity variable which depends on y and x , and f is the dimensionless stream function that depends on η only. The final transformed governing equation becomes

$$f''' + f f'' - \frac{2m}{m+1} f'^2 = 0 \quad (7)$$

subject to the following boundary conditions derived from equations (3), (4) and (6)

$$f'(0) = 1, f(0) = -v_w \sqrt{\frac{x^{1-m}}{\nu U_o}} \sqrt{\frac{2}{m+1}}, \quad f'(\infty) \rightarrow 0 \quad (8)$$

In deriving the second of the boundary conditions (8) the vertical injection or blowing speed v_w must be a function of the distance (for $m \neq 1$) from the leading edge. Consequently, v_w is described by the law

$$v_w = d U_w \text{Re}^{-1/2}, \quad \text{or } d = \frac{v_w}{U_w} \text{Re}^{1/2} \quad (9)$$

in order to have a similarity solution in which the mathematical solution contains η alone. It should be mentioned that v_w must be of order of magnitude of $U_w \text{Re}^{-1/2}$ [13], where $\text{Re} = U_w x/\nu$, in order to ensure that a flow with suction or blowing at the surface satisfies the assumptions of boundary layer theory. Therefore, d which is introduced as a blowing or suction parameter must be of order one (Bejan [14]). However, for $m = 1$, v_w is a constant does not contain x and the similarity solution still can

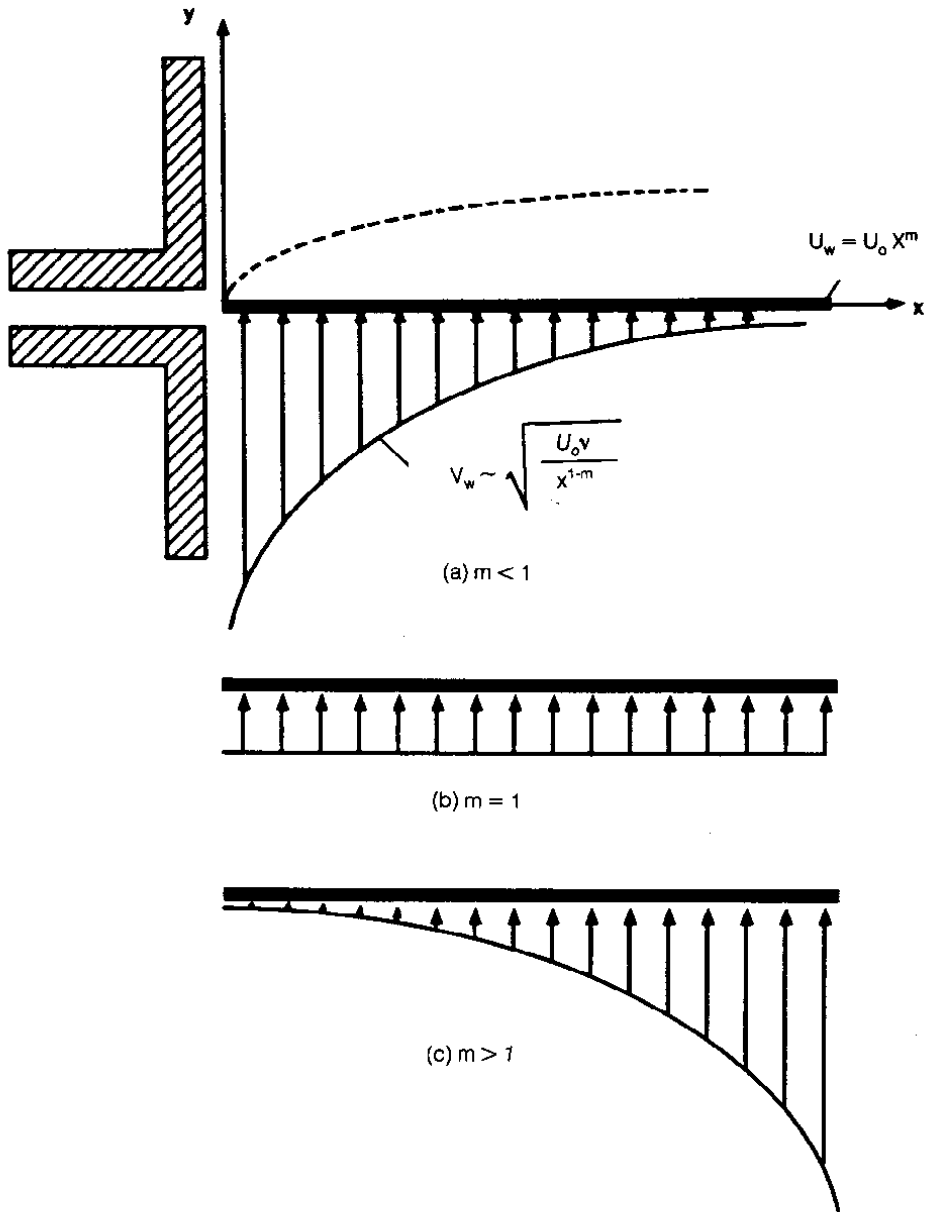


Fig. 1. Qualitative representation of boundary layer on a stretched surface with injection velocity v_w for (a) $m < 1$, (b) $m = 1$, (c) $m > 1$.

be found as described by Chow [15]. Equation (7) and the boundary conditions (8) reduced to Eqs. (8) and (9) in Gupta and Gupta [12] for $m = 1$. The second of the boundary conditions (8) can be written as

$$f(0) = -d\sqrt{\frac{2}{m+1}} \quad (10)$$

From Eq. (9), it is clear that suction or blowing parameter d is used to control the strength and direction of the normal flow at the boundary. Therefore, for positive or negative d we have a blowing or suction boundary condition respectively.

Expressions for shear stress and boundary layer thickness can be developed from the similarity solution of Eq. (7) and from Eq. (5) respectively

$$\tau = \mu U_o x^{m-1} \sqrt{\frac{m+1}{2}} \sqrt{\text{Re}_x} f''(\eta) \quad (11)$$

$$\delta = \frac{x}{\sqrt{\text{Re}_x}} \frac{\eta_{\max}}{\sqrt{\frac{m+1}{2}}} \quad (12)$$

In Eq. (11) $f'' \sqrt{((m+1)/2)}$ describes the dimensionless shear stress distribution in the boundary layer and the particular value of $f'' \sqrt{((m+1)/2)}$ at $\eta = 0$ represents the dimensionless shear stress on the stretching surface.

Numerical solution procedure

The fourth order Runge-Kutta-Merson method was used to integrate equation (7) subject to boundary conditions (8) and the modified $f(0)$ given by (10). The half-interval method is used to find $f''(0)$, and the algorithm has been modified following Chow [15]. The computation was carried out on an IBM compatible 386 personal computer machine. The solution provides f , f' , and f'' which depend on η_∞ . The value of η_∞ was chosen as large as possible between 7 to 20, depending on m and d without causing numerical oscillations in the solution.

Results and Discussion

Equation (7) subject to the first and third boundary conditions of (8) and the modified boundary condition (10) has been solved for the velocity parameter

$-0.93416 \leq m \leq 5.0$ and for different values of the dimensionless speed $-1.0 \leq d \leq 4.0$.

Some representative velocity profiles are presented in Fig. 2 for various values of m and for injection flow at the boundary for positive value of $d = 0.6$. In this figure the velocity profiles for $m > \phi$ exhibit exponential decay with no inflection points. However, for $m = -0.147$ the slope of the velocity profile is almost zero at $\eta = 0$ ($f'(0) \approx 0.005$) and it has a separation point. Furthermore, if m decreases a backflow develops and then for more negative values of m , a velocity overshoot is expected as shown in Fig. 3 for various negative values of m and in this case the surface is stretched with a decreasing velocity creating a decelerated flow. Moreover, in Fig. 3 the boundary layer thickness decreases with decreasing m and as m decreases the velocity profiles change their slopes to be steeper near the edge of the surface and the velocity overshoot stream continues. Comparing this case with decelerated flow over a stationary flat plate (Schlichting [13]) it can be seen that applying injection at the boundary of a decelerated surface increases the strength of the reverse or overshoot flow and reduces its boundary layer thickness.

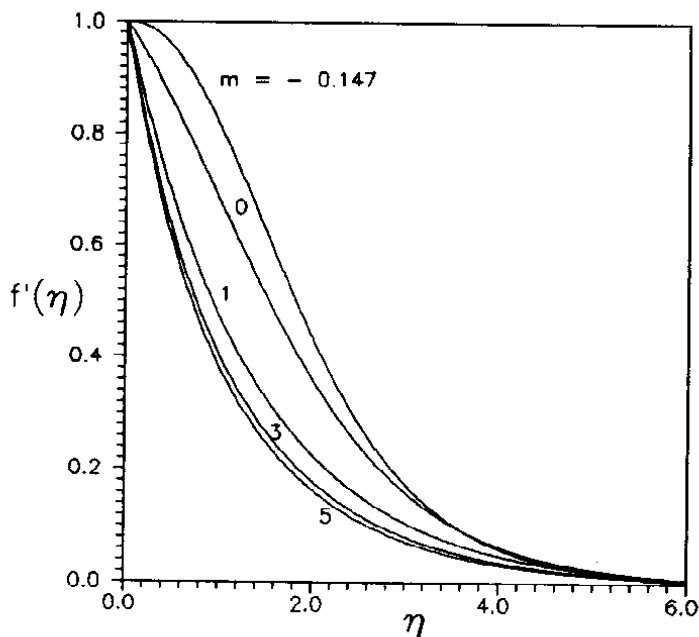


Fig. 2. Similarity velocity profiles as a function of the similarity variable η for various values of m and for $d = 0.6$.

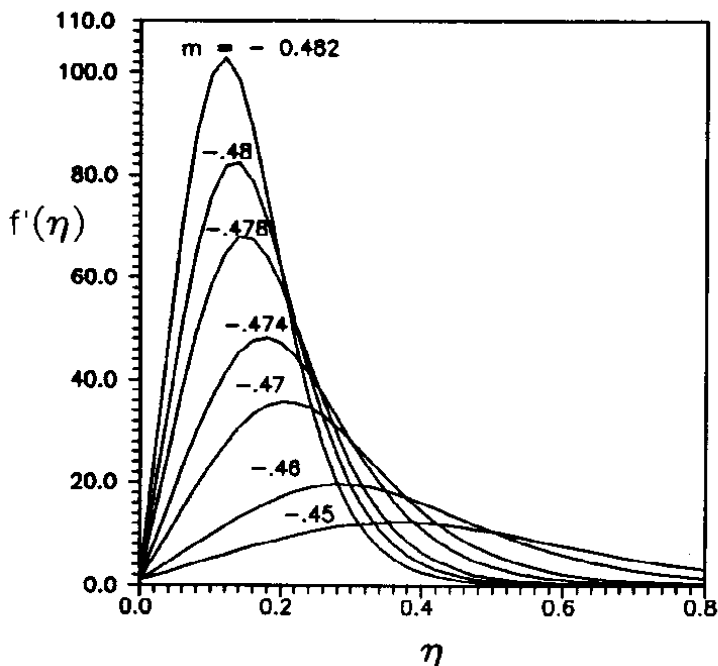


Fig. 3. Similarity velocity profiles indicating the velocity overshoot stream for negative values of m and for $d = 0.6$.

The presence of an inflection point in the velocity profile is an indication of flow instability (Lin [16]) and when the mass flow in blowing is made large, it is observed that the corresponding numerical calculation becomes difficult, because the velocity profile acquires a kink. Therefore, difficulties are developed in the computation for large negative values of m with positive values of d .

During the search for the similarity solutions of Eq. (7) subject to (8) and (10), the only accepted solution was the one in which the velocity profiles exhibit asymptotic decay to zero. Therefore, if the velocity profiles contain negative regions, in spite of these solutions satisfying the governing equation and its boundary conditions, they are unrealistic and are not considered.

The existence of non-unique solution has been found. Figure 4 presents $f'(\eta)$ as a function of the similarity variable η for $m = 2.0$ and 3.5 and for $d = 0.6$. In this figure the non-unique solutions of the velocity profiles are presented, each group of them exhibits one asymptotic decay solution and the other has a region of negative velocity for the same value of m . Similar curves of the velocity profiles for $d = 0.4$ are presented in Figs. 5 and 6 for various values of m .

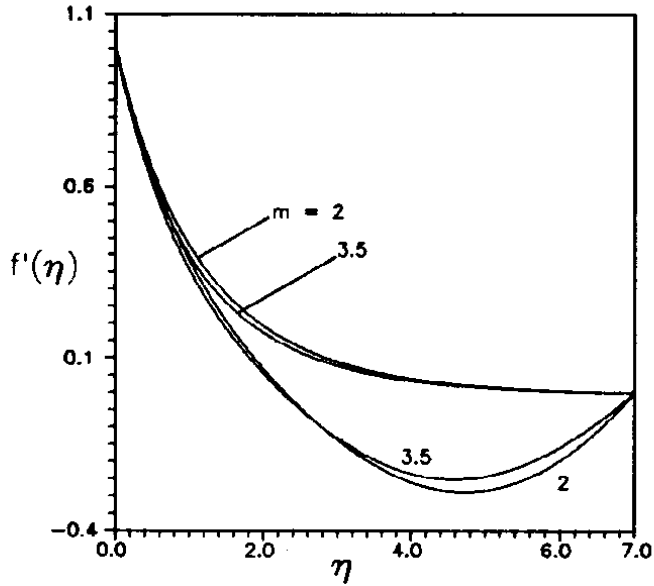


Fig. 4. Similarity velocity distributions showing the existence of non-unique solutions for $m = 2$ and 3.5 for $d = 0.6$.

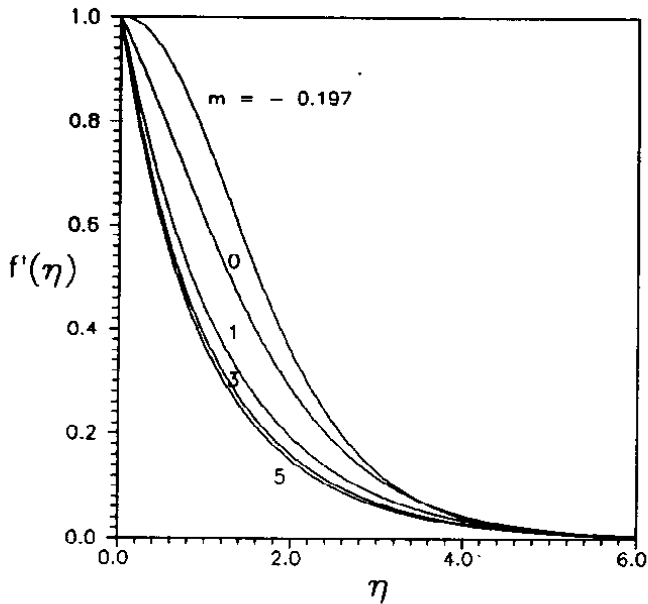


Fig. 5. Similarity velocity profiles as a function of η for different values of m at $d = 0.4$.

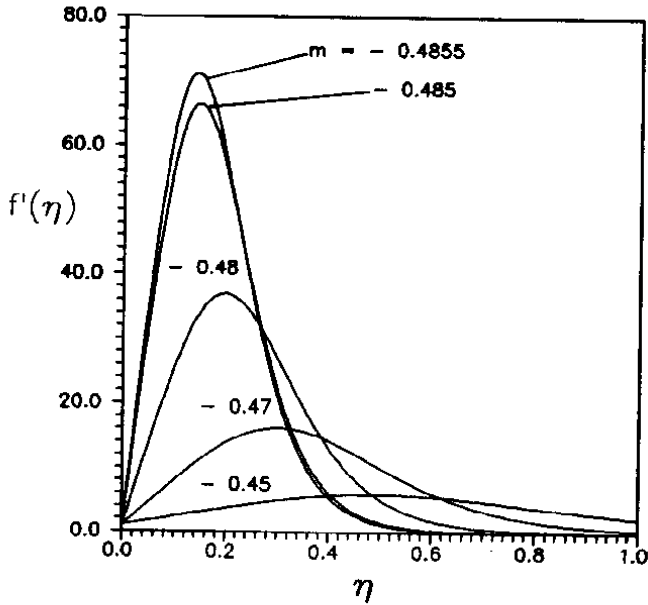


Fig. 6. Similarity velocity profiles showing the velocity overshoot flow for negative values of m and for $d = 0.4$.

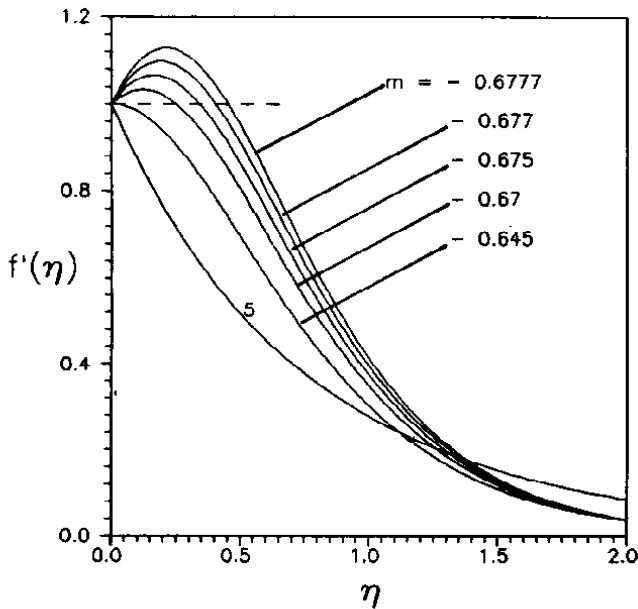


Fig. 7. Similarity velocity profiles for various values of m at $d = -0.6$, the dashed line indicating the backflow for $f'(\eta) > 1$.

The suction of the boundary layer from the stretched surface is shown in Fig. 7 for $d = -0.6$ and for different values of m . In this figure all the velocity profiles for $-0.645 < m \leq 5.0$ have asymptotic decay with no inflection point in the fluid. However, for $-0.645 \geq m$ the fluid has an inflection point and the reverse flow occurs above the dashed line and the separation point is very close to $m = -0.645$ where $f''(0) \approx 0.006$.

Comparing Figs. 3,6 and 7 it is clear that decreasing the controlling parameter d delays the reverse flow; in other words, m decreases with decreasing d and hence more asymptotic type solutions can be obtained with no inflection points. Table 1 shows the critical values of m , d for no reverse flow in the fluid. The relation between the critical values of m and d presented in Table 1 for zero surface shear is plotted in Fig. 8. The following third order polynomial is used to fit the data

Table 1. The critical values of the velocity parameter m and the injection parameter d for zero surface shear

d	m
2.2000	-0.0100
2.0000	-0.0120
1.8000	-0.0170
1.7000	-0.0190
1.6500	-0.0220
1.6000	-0.0230
1.5500	-0.0250
1.5000	-0.0280
1.4500	-0.0300
1.4000	-0.0320
1.2000	-0.0480
1.0000	-0.0750
0.8000	-0.1070
0.6000	-0.1470
0.4000	-0.1970
0.2000	-0.2600
0.1000	-0.2950
0.0000	-0.333335
-0.1532	-0.4000
-0.2000	-0.4220
-0.3158	-0.4800
-0.4000	-0.5255
-0.6000	-0.6446
-0.8000	-0.7805
-0.9000	-0.85506
-1.0000	-0.93414

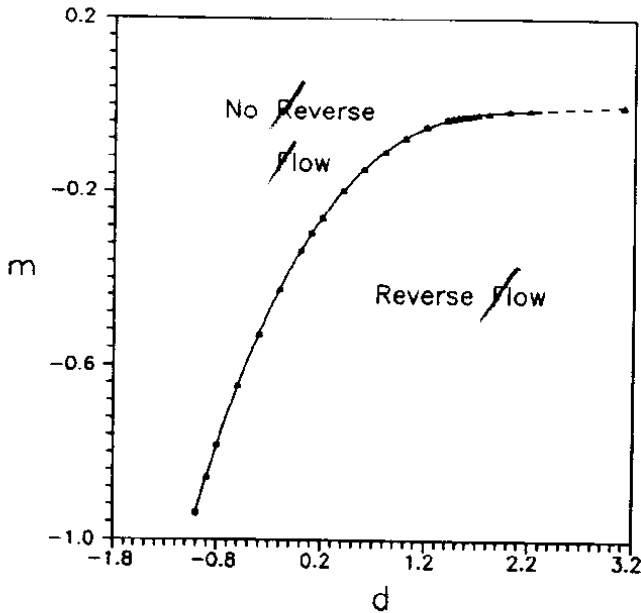


Fig. 8. The critical values of m as a function of the blowing parameter d , the dashed line connecting the computed points to the extrapolated one.

$$m = A + B d + C d^2 + E d^3$$

where A, B, C and E are constants given in the nomenclature. The last point in Fig. 8 is an extrapolated point which gives $d = 3.097$ for $m = -9.646e-5$ where m approaches zero from the negative side since for $m \geq 0$ all solutions have no reverse flow. Figure 8 shows that using suction on the stretched surface boundary delays the occurrence of reverse flow. It should be noted that for $m = -1$ there is no solution where $f(0)$ will blow up.

The nondimensional shear stress at the stretched surface presented by $((m+1)/2)^{1/2} f'(0)$ is shown in Fig. 9 for various values of m as a function of the dimensionless velocity d . In this figure the shear stress at the wall increases with increasing d and the velocity profiles corresponding to that change from asymptotic decay before $f'(0)$ reaches zero to profiles have inflection points for all positive values of $f'(0)$. Furthermore, $f'(0)$ reaches its maximum for $m = -0.2$ and -0.3 at a specific value of d , where there is a balance between the shearing force and the inertia force of injection, and beyond this value it decreases with increasing the injection parameter d .

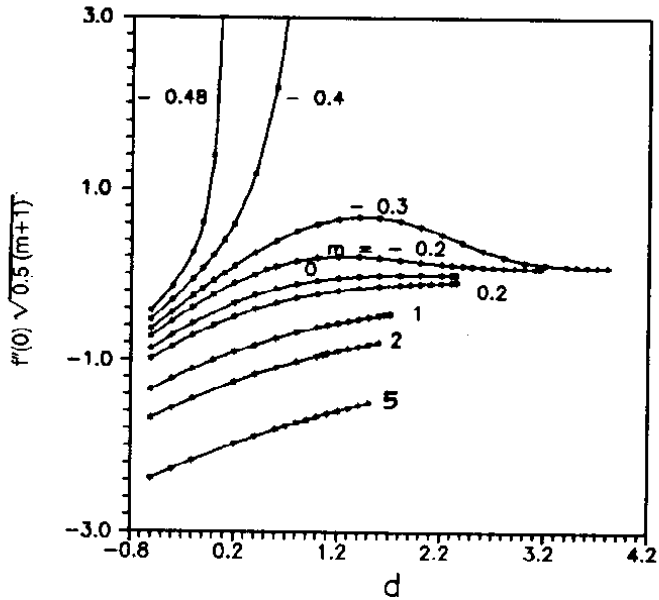


Fig. 9. The dimensionless shear stress distributions at the stretched surface boundary as a function of the blowing parameter d for various values of m .

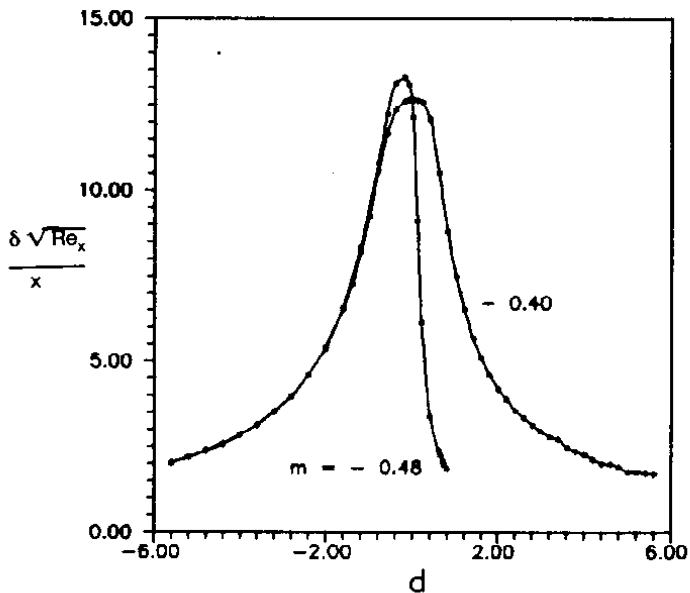


Fig. 10. The nondimensional boundary layer thickness distribution as a function of d for $m = -0.4$ and -0.48 .

Figure 10 presents the dimensionless boundary layer thickness defined by Eq. (12) as a function of d for $m = -0.48$ and -0.40 . In general increase in suction causes progressive thinning of the boundary layer for all m while reverse is true for injection. However, caution must be taken for negative m . If blowing is applied or suction degree is reduced ($d > d$ at $f''(0,0) = 0$) beyond the point of separation where $f''(0,0) = 0$, then the strength of backflow is built up. Furthermore, this is corresponding to decrease in δ up to a maximum value where a velocity overshoot. Moreover, if suction is allowed to increase or degree of blowing decreased ($d < d$ at $f''(0,0) = 0$) then δ decreases as in the general scenario mentioned before.

Summary and Conclusions

The flow field of the laminar boundary layer over a stretched surface that moves with a power law velocity profile with suction or injection at the boundary is reported for various values of m and d . It was found that for $m \geq 0$ and for all values of d , studied in this paper, the similarity solutions exhibit asymptotic behavior. However, for negative values of m , where the stretched surface decelerate in the x -direction, the backflow (or overshoot) is developed and its strength depends on d . Furthermore, if blowing is admitted ($d > 0$) at the surface boundary the strength of the reverse (or overshoot) flow increases with decreasing m for constant $d > 0$. Moreover, if suction is allowed ($d < 0$) at the boundary then the strength of backflow decreases sharply compared to the blowing case for the same negative values of m .

Non-unique similarity solutions have been found for the governing equation subject to the boundary conditions. Despite these non-unique solutions satisfying the governing differential equation and the boundary conditions, the only physically acceptable solution is the one exhibiting an asymptotic decay to zero.

It has been found that the dimensionless shear stress at the surface increases for increasing d for $m \geq 0$ and it asymptotically decays to zero as for $m = 0$, and 0.2 . Furthermore, the shear stress continues to increase up to a maximum value that depends on d and m ($m < 0$) and then decreases asymptotically to zero as seen for $m = -0.2$ and -0.3 . The boundary layer thickness decreases when suction or injection is applied for $m = -0.4$ and -0.48 .

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تأثير الشفط أو النفخ على تطور التدفق الطبقي للطبقة الجدارية فوق سطح يسحب

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ملخص البحث. تم تقديم حلول مشابهة للتدفق الانسيابي للطبقة الجدارية فوق سطح أفقي مستمر السحب مع وجود شفط أو نفخ على السطح. فقد فرض أن هذا السطح يتحرك حسب قانون العلاقة الأسية للسرعة حيث كانت قيمة ثابت أس السرعة m أكبر من أو تساوي -0.93416 ، وأصغر من أو تساوي 0 كما أن هذا السطح معرض لمعامل تحكم في الشفط أو النفخ عليه d أكبر من أو تساوي -1 وأقل من أو تساوي 4 . وقد وجد أن الشفط للطبقة الجدارية على السطح المسحوب يؤخر التدفق العكسي بينما النفخ يزيد من شدة التدفق العكسي (أو زيادة السرعة زيادة كبيرة) كلما زادت القيم السالبة لـ m . كما وجد أيضاً لقيم m المختلفة حلول للتدفق غير وحيدة لمعادلات التحكم لنفس الشروط الحدودية. وعلاوة على ذلك فقد وجد لبعض أنواع التدفق بعجلة تقصيرية (الذي فيه m أصغر من الصفر) أن إجهاد القص عديم الأبعاد عند السطح المسحوب يزيد مع زيادة d إلى أقصى قيمة له حيث هناك اتزان بين قوة القص وقوة القصور الذاتي للنفخ وبعد ذلك يقل تدريجياً إلى الصفر. ومن الناحية الأخرى فإنه لتدفق بعجلة تزايدية (فيه m أكبر من الصفر) فإن إجهاد القص عديم الأبعاد يزيد تدريجياً إلى الصفر.