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# Modeling GPS-Derived Orthometric Heights for a Small and Dense Network

**Abdullah S. Alsalman**
*Civil Engineering Department, College of Engineering*
*King Saud University, P.O. Box 800, Riyadh 11421, Saudi Arabia*

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**Abstract.** The Global Positioning System (GPS) relative carrier phase observations would provide a powerful and fast tool for obtaining an accurate relative three dimensional positioning of a point on the earth. However, the orthometric heights (elevations above geoid), which are the required values for most of surveying and engineering applications, are not directly provided by GPS observations. Instead, the ellipsoidal heights (elevations above ellipsoid) are derived from the processing of GPS observations. It follows that the separation between the ellipsoid and geoid (geoidal height) is an essential quantity in determining orthometric heights using GPS observations. This paper presents an accurate, relatively simple, and reliable technique to precisely model geoidal heights by combining both geopotential and regression surface fitting approaches utilizing least squares method for each one.

For the purpose of this proposed approach, a dense and small test network was designed. Both precise leveling and GPS observations were carried out and adjusted for common stations to establish precise vertical control network. Then, the above mentioned technique was implemented and a geoidal model for the area was developed and tested. The results based on the developed model showed that GPS-derived orthometric heights were better than 1 cm, and a third order leveling specifications were achievable without differential leveling.

## Introduction

The Global Positioning System (GPS) can provide ellipsoidal height differences with a precision approaching $\pm$ (2 mm + 0.01 - 0.1 ppm) [1]. Additionally, the GPS technology can provide this level of precision in few minutes, without requiring the intervisibility of the stations. One drawback is that the GPS does not provide a direct method for determining orthometric heights (i.e. heights above geoid) of newly established stations because it is a pure geometric positioning system. This means that GPS can provide ellipsoidal heights. However, orthometric heights are required for most engineering works since orthometric height differences reflect the potential for, and direction of fluid flow.

With the growing use of GPS technology several publications deal with theories, methods and practical experiences that are available for determining orthometric heights from GPS observations [1-5]. However, some of the previously published reports use only geopotential (i.e. gravity-based) models to obtain the geoid for the area of interest while others may apply pure geometric methods to interpolate the geoid. Both approaches may result in uncertainties of geoid height difference estimates when used independently. Therefore, combining the two approaches would provide more accurate information about the geoidal height (i.e. separation between geoid and ellipsoid); hence, it offers more accurate solution to the problem. However, the basic requirement for applying this approach is to have accurate vertical control stations that have been occupied by accurate relative carrier phase GPS observations.

The aim of this paper, therefore, is to present a procedure that has been performed to establish a precise vertical control test network within King Saud University Campus in Riyadh, Saudi Arabia. Next, use GPS techniques to obtain the geodetic positions (latitude $\phi$, longitude $\lambda$, and ellipsoidal height h) at locations of vertical control stations. Then, compute the geoidal heights using global geopotential models for the same control stations. Finally, develop best fitting local geoid surface model for the area using the established vertical control stations.

## Description of Test Network

King Saud University main campus north of Riyadh city extends over roughly 15 square kilometers. It is desirable to use GPS technology to obtain detailed topographic maps and digital elevation models for the campus to satisfy surveying and engineering projects and to support Geographical Information System (GIS) requirements within the campus. However, as mentioned earlier, there should be an easy, accurate and economic procedure to provide the orthometric heights from GPS measurements. Therefore, sixteen stations were selected around the perimeter and the center of the campus. The location of the stations were designed to be a suitable site for GPS observations in terms of satellite visibility, potential multipath, and accessibility as well as good representation of topography. One of the selected stations (station 13) is an existing first order National Geodetic Network (NGN) station (see Fig. 1). The design of such dense network would make it an ideal site for study of the use of GPS to derive orthometric heights; and hence replacing the tedious, time consuming, and costly classical differential leveling required for engineering and surveying applications.

The essential quantity that enables conversion of a GPS ellipsoidal height (h) to an orthometric height (H) is the geoidal height (N). For this reason, it is necessary that geoidal heights be computable from known values of ellipsoidal and orthometric heights at these test network stations. The first quantity can be provided by GPS observations while the second one must be obtained from differential leveling.

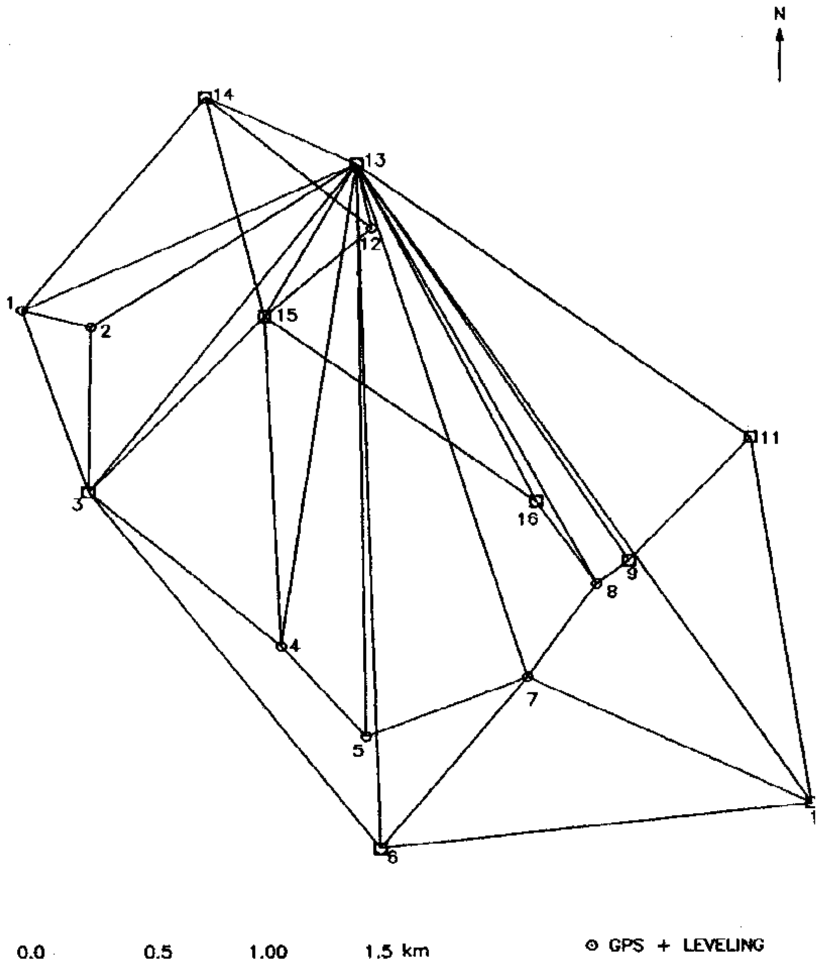


**Fig. 1. Layout of GPS test network.**

## GPS Observations

The first order GPS campaign has been performed to establish geodetic positions of the selected stations in the test network. All sixteen stations were occupied, in static mode, by Leica SR 9500, dual frequency, with twelve chanels receivers. The schedule of observations was designed to conform to U.S.A. Federal Geodetic Control Committee (FGCC) specifications corresponding to first order GPS standards [6]. The executed observation sessions have been processed with the Leica-SKI post processing software [7]. Then, the GPS coordinates ($\phi$, $\lambda$, h) were obtained from minimally constrained least squares adjustment holding the NGN station (station No. 13) fixed.

It should be pointed out here that this article is not intended to elaborate on the subject of processing and operation of the Global Positioning System, nor the use and adjustment of geodetic network in GPS surveys. These subjects can be found in many published articles and books [8-10]. However, the results of these processes are essential in order to perform the research in hand. The obtained results showed high internal accuracy (see Table 1); this high accuracy was expected due to the relatively short baselines and good planning for the maximum number and best distribution of GPS satellites during the observation periods.

**Table 1. Results of GPS network adjustment**

| Station no. | $\sigma\phi$ (mm) | $\sigma_\lambda$ (mm) | $\sigma_h$ (mm) |
|---|---|---|---|
| 1 | ±3.2 | ±2.9 | ±3.5 |
| 2 | ±3.1 | ±3.2 | ±3.8 |
| 3 | ±3.6 | ±3.2 | ±3.7 |
| 4 | ±3.9 | ±3.6 | ±3.8 |
| 5 | ±5.1 | ±4.5 | ±5.3 |
| 6 | ±4.8 | ±3.7 | ±4.0 |
| 7 | ±4.6 | ±4.5 | ±4.5 |
| 8 | ±4.1 | ±4.2 | ±4.9 |
| 9 | ±3.8 | ±3.4 | ±5.2 |
| 10 | ±5.4 | ±5.1 | ±5.8 |
| 11 | ±4.3 | ±4.4 | ±4.5 |
| 12 | ±2.1 | ±2.6 | ±3.1 |
| 13 | ±0.0 | ±0.0 | ±0.0 |
| 14 | ±2.4 | ±2.2 | ±2.9 |
| 15 | ±3.0 | ±3.2 | ±3.3 |
| 16 | ±4.4 | ±4.3 | ±4.1 |

Since the baselines vectors obtained from GPS observations are usually referenced to World Geodetic System 1984 (WGS84), while the mapping system in the Kingdom of Saudi Arabia utilizes Ain-Elabd 1970 datum, which is called NGN datum [11], it is necessary to perform a three dimensional transformation from WGS84 datum, to NGN datum. The transformation parameters were given by [12]. The last step is to convert the final results into Universal Transverse Mercator projection grid system (UTM). The UTM coordinates system is the one currently used for horizontal mapping purposes in the Kingdom.

## Differential Leveling

As mentioned earlier, establishment of a precise vertical control stations at the locations of GPS control stations is a necessary step in order to proceed with the estimation of geoidal heights using the proposed technique within the area. However only one of the sixteen stations (No. 13) has reliable orthometric height available. Therefore, a precise differential leveling project was conducted. Since it is desirable to obtain centimeter level geoidal heights, a relative accuracy to few millimeter range was sought for the orthometric heights to be determined by differential leveling. This can be accomplished by satisfying the USA based FGCC first order, class I leveling specification [13]; thus the allowable misclosure of a level loop can be given as;

$$e = \pm 4 \text{ mm} \sqrt{K}$$

where

| | | |
|---|---|---|
| $e$ | = allowable misclosure of a level circuit in mm. |
| $K$ | = the leveled length of the loop in Km. |

A recently calibrated Wild N3 precise level equipped with a parallel plate micrometer was used for all leveling observations. The micrometer has a range of one cm, a least count of one mm, and permits estimation to 0.1 mm. A Wild GPLE3 precise leveling staff, with double scale, invar strip faces, and one cm. divisions was used.

The fieldwork has been conducted with doublerun, with both high and low scale readings taken for the sighting at each setup. The obtained raw data were reduced by computing the average of low and high scale observations. The difference of backsight and foresight distances should not exceed 2 m per set-up. The requirement is needed to minimize collimation error, curvature and refraction effects. Maximum sight distance was limited to 50 m; and minimum ground clearance of line of sight was larger than 0.5 m. All observations were taken in either early morning or late afternoon. This would help in reducing the effect of heat waves and shimmering during the observation process [14]. Furthermore, the average temperatures during the observations periods were recorded to ensure the small variation in them and hence stop the observations process when these changes are significant.

A misclosure was computed for each loop in the network. All misclosures were within the acceptance ranges according to first order class I specifications (i.e. 4 mm $\sqrt{K}$). About

25 Km of leveling lines were doublerun, connecting the sixteen stations of the test network.

The final output of the precise differential leveling is the orthometric heights of the fifteen stations relative to the known NGN benchmark. The results are shown in Table 2.

**Table 2. Results of level network adjustment**

| Station no. | $\sigma_H$ (mm) | Station no. | $\sigma_H$ (mm) |
|---|---|---|---|
| 1 | ±3.3 | 9 | ±4.8 |
| 2 | ±3.4 | 10 | ±5.9 |
| 3 | ±4.0 | 11 | ±4.8 |
| 4 | ±4.2 | 12 | ±2.3 |
| 5 | ±5.1 | 13 | ±0.0 |
| 6 | ±5.8 | 14 | ±3.3 |
| 7 | ±5.7 | 15 | ±3.4 |
| 8 | ±4.6 | 16 | ±4.8 |

## Geoidal Heights from Geopotential Models

For the purpose of this research, the OSU91A geopotential model, developed by Prof. R. Rapp at the Ohio State University, was utilized. This is a global geopotential model using spherical harmonics complete to degree and order 360 [15]. It was determined using a set of 30-minutes of latitude and longitude mean gravity anomalies, derived from satellite altimetry data collected over the ocean areas and from land measurements in North America, Europe, Australia, Japan, Asia and Africa. The input data for OSU91A model are the geodetic latitude and longitude computed from GPS, for each station.

The computed geoid values estimated from the OSU91A model were used in the subsequent processing for the model development.

## Mathematical Models Formulation

The approximate relationship between the ellipsoidal height and the orthometric height may be expressed as follows:

$$h \approx H + N \tag{1}$$

where

h = ellipsoidal height;
H = orthometric height; and
N = geoidal height.

Equation 1 is not an exact relationship since H is the distance along the plump line while h is normal to the ellipsoidal. The angular difference between the plump line and the normal is the deflection of the vertical. However, the error introduced by neglecting the deflection of the vertical in Eq. 1 is insignificant when compared with the uncertainties in geoidal height difference estimates and orthometric height differences [1], [9, p. 264].

The first step of the proposed approach of modeling GPS-derived orthometric heights is to estimate the best geoidal height differences. Using Eq. 1, assuming that the adjusted carrier phase GPS observations and the adjusted orthometric heights from precise differential leveling of existing control network are error-free. These assumptions are reasonable since the ellipsoidal height differences were determined from GPS relative carrier phase observations with accuracy better than 1 ppm. Also, the orthometric height differences were obtained by precise differential leveling that conforms to first order class I FGCC specifications. The computed values from the geopotential model are considered as observations for the geoidal heights. Thus a least squares adjustment may be applied to obtain a refined geoidal heights that accurately represent the local relief of the geoid in the area of study by adding appropriate correction terms.

The observation equation for a station i representing the functional relationship, then, becomes:

$$h_i = H_i + N_i + dN_i \tag{2}$$

where

$dN_i$ = correction for geoidal height at control station i.

Equation (2) may be rearranged as:

$$H_i + N_i + dN_i - h_i = 0 \tag{3}$$

Other observation equations similar to Eq. 3 can be written for each station in the network. Hence, for any two stations i and j, an equation relating the height differences between them, may be written as:

$$H_i - H_j + N_i - N_j + dN_i - dN_j - h_i + h_j = 0 \tag{4}$$

Equation (4) conforms with the condition equations approach in which the mathematical model consists of conditions between the observed quantities (i.e., $N_i$ and $N_j$).

Each pair of stations may form an equation similar to (4) forming a total of 15 independent equations for this particular network. These equations can be expressed in matrix form as follows:

$$B_{(m-1)xm} \ V_{mx1} + C_{(m-1)x1} = O_{(m-1)x1} \tag{5}$$

where

B = the design matrix for the observations.
V = the vector of residuals (or corrections to geoidal heights).
C = the vector of misclosure.
m = the number of observations (control stations).

Solving set of equations 5 yields the corrections to geoidal heights. These corrected geoidal heights along with the UTM grid coordinates for the control stations were used, as the second step in this approach, to develop a three-dimensional regression surface that best describes the local geoid within the network.

## Local Geoidal Surface Models

The selection of best-fitting surface equation clearly depends on many factors, such as the size of network, undulation of terrain, number and distribution of control stations to be used, and the quality of GPS and leveling observations. Most of these factors were considered during the design of the network. Least squares stepwise regression analyses were performed using MINITAB statistical software package as well as MATLAB software. These software packages were used only as a tool for deriving the best-fitting geoid surface for project area. Therefore, it is not the purpose of this paper to elaborate on how to use them, but rather to present their results.

Conceptually, the three-dimensional geoidal surface model that best represents the real surface of the area may be expressed as [16]:

$$N = \sum_{i=0}^{n} \sum_{j=0}^{n} a_{ij} X^i Y^j \tag{6}$$

where

N = adjusted geoidal height;
$a_{ij}$ = estimated unknown parameters;
X,Y = UTM grid coordinates of control stations; and
n = order of polynomial.

For the test network utilized in this research, (i.e., 16 stations) the maximum number of unknown parameters to be used can not exceed 16 (i.e. third order polynomial that has 16 unknown parameters). However, for this small size area and relatively smooth terrain, it was decided to start with biquadratic surface of the type defined in Eq. 6. In addition, it was decided to utilize only 9 of these established control stations (i.e. stations 3,6,9,10,11,13,14,15, and 16) to fit the local geoid, and use the remaining 7 stations (i.e. stations 1,2,4,5,7,8, and 12) as check stations to test the goodness of fit.

Therefore, second order polynomial (i.e. 9 unknown parameters) was examined parameter-by-parameter in a least-squares step-wise regression fashion until the best-fitting geoidal surface was obtained.

The developed geoidal model may be expressed as follows:

$$N = a_{00} + a_{10} X + a_{01} Y + a_{11} XY \tag{7}$$

where

$a_{00}$ = 17.1667
$a_{10}$ = $-2.86828 * 10^{-5}$
$a_{01}$ = $-2.70643 * 10^{-5}$
$a_{11}$ = $7.90541 * 10^{-9}$

Table 3 shows the residuals of the regression Eq. 7 found to best represent the local geoid for the project area. Also, Table 4 shows the estimated orthometric heights versus known orthometric heights for control stations that were not used in the surface fitting. It is to be pointed out that the computed variations in geoidal heights were about ± 5 cm in the test area.

**Table 3. Results of orthometric height differences for control stations used for surface fitting**

| Station no. | $\sigma_H$ (mm) | Station no. | $\sigma_H$ (mm) |
|---|---|---|---|
| 3 | -8.2 | 13 | -9.8 |
| 6 | +3.0 | 14 | -7.8 |
| 9 | -2.5 | 15 | +8.6 |
| 10 | +0.3 | 16 | -4.1 |
| 11 | +1.8 | | |

**Table 4. Results of predicted orthometric heights for check control stations**

| Station no. | GPS-derived orthometric height (m) | Precise leveling orthometric height (m) | dH* (cm) |
|---|---|---|---|
| 1 | 666.131 | 666.140 | -0.9 |
| 2 | 651.128 | 651.120 | +0.8 |
| 4 | 640.955 | 640.956 | -0.1 |
| 5 | 640.917 | 640.912 | +0.5 |
| 7 | 643.923 | 643.927 | -0.4 |
| 8 | 650.943 | 650.942 | +0.1 |
| 12 | 673.299 | 673.290 | +0.9 |

*dH = 00.

GPS-derived orthometric height-precise leveling orthometric height.

## Discussion and Analysis

Tables 1 through 4 show the final outcome of this research project. These tables are largely self-explanatory. However, it is appropriate to add some statistics and comments that can be deduced from them. The network standard deviations of $\sigma\phi$, $\sigma\lambda$, and $\sigma_h$ obtained from GPS observations were ± 1.3 mm, and ± 1.2, and ± 1.3 mm, respectively, while the mean of these values were ± 3.6 mm, ± 3.4 mm, and ± 3.9 mm, respectively. The network standard deviation of the orthometric heights obtained from precise leveling was ± 1.5 mm, while the mean was ± 4.1 mm. These values would indicate a high quality of the leveling network. Thus, the assumptions of error-free ellipsoidal heights obtained from GPS measurements as well as the orthometric heights obtained from leveling were reasonable and valid.

The residuals obtained from the least squares surface fitting, shown in Table 3, reflects the appropriateness of the local geoid model and also reflects the accuracy of the input data used to establish the model. The mean and standard deviation of these residuals were -2.1 mm and ± 6.1 mm, respectively.

Finally, a further verification for the goodness of fit for stations with precisely known orthometric heights within the area was performed. As shown in Table 4, the maximum discrepancy between the predicted orthometric height and the known orthometric height was 0.9 cm, while the root mean square error (RMS) of these discrepancies was 0.62 cm.

Considering the previously mentioned mean error of ± 3.9 mm for ellipsoidal heights and ± 4.1 mm for leveling; the propagated error is ± 5.7 mm; this is in agreement with the RMS of the difference between the predicted and known orthometric heights (i.e. 6.2 mm). These results would also indicate the importance of accuracy and distribution of control station within the project area. Ideally, the geometry of these stations should be such that control stations are well-distributed around the perimeter and in the center of the project area. This means that extrapolation should be avoided.

Comparing these results with the USA-based FGCC specifications, it was found that the third order vertical standards can be easily achieved [13]. Thus, orthometric heights can be computed for any station within the area based on the data obtained from GPS observations and the geopotential model.

It should be pointed out, however, that this technique cannot be generalized into other larger regions without a careful study to the topography of the area of interest. Therefore, the results obtained in this experiment might be suitable for this particular area and can be applied only within a dense and small network in order to achieve cm level accuracies. Moreover, the procedure presented in this paper requires the knowledge of geoidal height for at least one station in the study area. Finally, the fact that the area of study has a relatively smooth terrain would make this procedure valid and applicable. Otherwise, the accuracy is largely degraded in areas which have abrupt change in elevation such as cliffs, wells, etc.

## Conclusions

It is clear that GPS-derived orthometric heights will have a major impact on the engineering and surveying community because of the feasibility and efficiency of this technique compared to leveling. The GPS-derived geoid model presented in this paper proved to be accurate and simple model to be implemented for a relatively small and smooth terrain area. It combined geopotential model together with geometric model in sequential least squares adjustment fashion to produce an acceptable level of desired third order vertical accuracy.

The proposed approach would reduce the cost, time, and effort required to perform differential leveling needed for surveyors and engineers. The results obtained from this research indicate that the achieved accuracies of the estimated orthometric heights are reliable and well above most of the large scale mapping requirements. This high quality, however, requires, as input, very accurate GPS ellipsoidal and leveling orthometric heights at some common stations.

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# بناء نموذج رياضي للارتفاعات الأرثومترية المشتقة من نظام التوقيع العالمي لشبكة صغيرة وكثيفة

**عبدالله سلمان السلمان**

*قسم الهندسة المدنية، كلية الهندسة، جامعة الملك سعود، ص ب ۸۰۰،*

*الرياض ١١٤٢١، المملكة العربية السعودية*

(أستلم في ١٩٩٧/٩/٣م؛ وقُبل للنشر في ١٩٩٧/١١/٥م)

**ملخص البحث.** تعطي أرصاد نظام التوقيع العالمي - المعروف إختصارا GPS - دقة نسبية عالية في التوقيع ثلاثي الأبعاد لجميع النقاط على سطح الكرة الأرضية وبسرعة فائقة، ولكن الارتفاعات الأرثومترية (الارتفاعات فوق سطح الجيوئيد) لا يمكن الحصول عليها مباشرة من نتائج أرصاد هذا النظام على الرغم من أنها هي التي تستخدم عادة في أعمال المساحة والتطبيقات الهندسية المترتبة عليها. والذي يتم الحصول عليها مباشرة هي الارتفاعات الأهليليجية (الارتفاعات فوق سطح القطع الناقص) لذلك لابد من معرفة الفرق بين سطح القطع الناقص والجيوئيد (الارتفاعات الجيوئيدية) عند استخدام نظام التوقيع العالمي.

يقدم هذا البحث طريقة عملية وسهلة وذات دقة وموثوقية عالية لإيجاد نموذج دقيق للارتفاعات الجيوئيدية باستخدام طريقة ضبط التربيعات الأقل لكل من الجهد الأرضي وسطح الملائمة الإحصائي. ولتحقيق الغرض من الدراسة المقترحة تم تصميم شبكة كثيفة ومصغرة من خلال عمل وضبط أرصاد لكل من أجهزة نظام التوقيع العالمي والميزانية الدقيقة لنفس النقاط، وقد تم تطبيق الطريقة المذكورة واختبارها على هذه الشبكة، وقد أعطى النموذج المطور في هذا البحث نتائج أدق من ١سم وتم تحقيق دقة ميزانية تسوية من المستوى الثالث وبدون اللجوء للطرق التقليدية للميزانية.