J. King Saud Univ., Vol. 9 (*Science*), 2, pp.211-220 (A.H. 1417/1997)

MATHEMATICS

Quartic Piecewise Approximation in Bernstein Form

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(Received 7 April ,1996; accepted for publication on 29 October 1996)

Abstract. We study and give the essential conditions on quartic piecewise approxmations in Bernstein form in order to get a high accuracy order. The symbolic language Maple is used to write a program to simplify the method and make it easy in the applications. It is shown that a 7^{th} order is attainable for quartic piecewise approximation in Bernstein basis.

Preliminaries

Parametric polynomial representations are used for free-form curve and surface design in Computer Aided Geometric Design (CAGD) systems. These are easily influenced by the types of polynomial bases, degree of accuracy, and the degree of the polynomial. The most frequently used bases are the Bernstein-Bézier basis, Schoenberg B-Spline basis, and the Hermite basis, or even the plain canonical basis for curves. Recently a considerable amount of research has been done on the problem of developing Bézier schemes.

Consider the data

$$S := \left\{ y_i^{(j)} ; i = 0, 1, \ldots, m+1, j = 0, 1, \ldots, n-1, \ \ m, n \in N \right\}$$

and let Δ define the set of nodes of interpolation

$$a = t_0 > t_1 < \ldots < t_m < t_{m+1} = b$$

and let P_{2n-1} be the set of polynomials of degree 2n -1. The classical problem in approximation theory of finding an interpolatory polynomial to S at the abscissae of Δ by a piecewise polynomial $p \in P_{2n-1}$ requires the satisfaction of

$$p^{(j)}(t_i) = y_i^{(j)}; i = 0,1,\ldots,m+1, j = 0,1,\ldots,n-1.$$

This is called the Hermite space $H^{(n)}(\Delta)$; defined by

$$H^{(n)}(\Delta) := \left\{p(x) \in C^{n-1}[a,b] : p(x) \in P_{2n-1} \text{ on each subinterval of } \Delta\right\}.$$

This linear space has the dimension n(m + 2). And thus $p \in H^{(n)}(\Delta)$ is defined by the associated n(m + 2) linear conditions. The existence and uniqueness can be shown by considering each subinterval $[t_i, t_{i+1}]$ on which 2n conditions determine uniquely a polynomial $p \in P_{2n-1}$. In the applications of CAGD the parameters t_i, t_{i+1} have to be determined from the geometric interpolation because only positions are given. In this paper the values $y_i^{(j)}$ are obtained from a given curve $C : t \mapsto (f(t), g(t))$, $t \in R$ and its derivatives, and thus are independent of changes of the parametrization.

Notations and Definitions

Definition 1: The α - Jet $(J_0^\alpha(C)$ of order $\alpha \in N$ of a curve C at t = 0 is $(C(0), C'(0), C''(0), \ldots, C^{(\alpha)}(0))^T$.

Definition 2: Two regular curves C and $P : t \mapsto (X(t), Y(t))$, $t \in R$ agree at t = 0 with order $\alpha \in N$ if there exists a parametrization $\varphi : R \mapsto R$ with $\varphi(0) = 0$ and $J_0^\alpha(C) = J_0^\alpha(P)$.

Without loss of generality we may assume that C(0) : P(0) : = (0,0), and C'(0) : = (1,0). Thus C can be parameterized in the form C $t \mapsto X(t) \mapsto (X(t), g \circ f^{-1} \circ X(t))$, $t \in R$ and thus the above definition is equivalent to the following condition

$$g \circ f^{-1} \circ X(t) - Y(t) = \vartheta(t^\alpha)$$

where o denotes the composition of functions. Setting $F(t) := g \circ f^{-1} \circ X(t)$, then P approximates C in [t1,t2] with order 8 iff

$$\left(\frac{d}{dt}\right)^j \{F(t)-Y(t)\}_{|t=t1} = 0, \quad j=0,1,2,3,$$

$$\left(\frac{d}{dt}\right)^j \{F(t)-Y(t)\}_{|t=t2} = 0, \quad j=0,1,2,3, \tag{1}$$

and

$$X(t1)=t1, \quad X(t2)=t2.$$

Here we take X(t) and Y(t) to be the quartic Bernstein polynomials defined by

$$X(t) := a_0\, B_0^4(t) + a_1\, B_1^4(t) + a_2\, B_2^4(t) + a_3\, B_3^4(t) + a_4\, B_4^4(t)$$

$$y(t) := b_0\, B_0^4(t) + b_1\, B_1^4(t) + b_2\, B_2^4(t) + b_3\, B_3^4(t) + b_4\, B_4^4(t)$$

where $B_i^4(t) = \binom{4}{i}(1-t)^{4-i}\, t^i\,; i=0,\ldots,4$. There are 10 free parameters for X(t), and Y(t), and there are 10 equations in the system of equations in (1), which means that C can be approximated by a piecewise polynomial curve of degree 4 with order 8.

For simplicity from now on we set t1 : = 0 and t2 : = 1 and use the abbreviations:

$$F_{i+1,j+1} := \left(\frac{d}{dt}\right)^j F(i)\,;\; i=0,1, \quad j=0,1,2,3 \tag{2}$$

$$x_{i+1,j+1} := \left(\frac{d}{dt}\right)^j X(i)\,;\; i=0,1, \quad j=0,1,2,3 \tag{3}$$

$$y_{i+1,j+1} := \left(\frac{d}{dt}\right)^j Y(i)\,;\; i=0,1, \quad j=0,1,2,3 \tag{4}$$

Sufficient Conditions

Applying the conditions in (1) and using the abbreviations in (2), (3), and (4) we get the following associated nonlinear system of equations:

$$e1 := 0 - x_{1,1}$$
$$e2 := F_{1,1} - y_{1,1}$$
$$e3 := F_{1,2}\, x_{1,2} - y_{1,2}$$
$$e4 := F_{1,3}\, x_{1,2}^2 + F_{1,2}\, x_{1,3} - y_{1,3}$$
$$e5 := F_{1,4}\, x_{1,2}^3 + 3F_{1,3}\, x_{1,2}\, x_{1,3} + F_{1,2}\, x_{1,4} - y_{1,4}$$

$$e6 := 1 - x_{2,1}$$
$$e7 := F_{2,1} - y_{2,1}$$
$$e8 := F_{2,2}\, x_{2,2} - y_{2,2}$$
$$e9 := F_{2,3}\, x_{2,2}^2 + F_{2,2}\, x_{2,3} - y_{2,3}$$
$$e10 := F_{2,4}\, x_{2,2}^3 + 3F_{2,3}\, x_{2,2}\, x_{2,3} + F_{2,2}\, x_{2,4} - y_{2,4}.$$

Lemma: An 8th order of approximation would be attained by solving:

$$e1 = e2 = e3 = e4 = e5 = e6 = e7 = e8 = e9 = e10 = 0.$$

Restrictions and Solutions

Since $e4 = 0$ and $e9 = 0$ are quadratic equations, and $e5 = 0$ and $e10$ are cubic equations, it is not possible to solve this nonlinear system of equations. One choice is to set $x_{1,2} :=$ constant $:=$ c to simplify the nonlinear system and then find a solution. This restriction reduces the order of approximation to 7. This assumption requires the replacement of e10 by e0 below. To simplify the notation, set

$$A := 48\, F_{2,3}\, F_{1,3}\, c$$

$$B := \left(-96\, F_{1,3}\, F_{2,3}\, c + 24\, F_{1,2}^2 - 48\, F_{1,2}\, F_{2,2} + 24\, F_{2,2}^2\right)$$

$$D := -12F_{1,2}\, F_{1,3}\, c^2 - 36F_{1,1}\, F_{1,3}\, c + 48F_{1,3}\, F_{2,3}\, c - F_{2,2}\, F_{1,4}\, c^3 + 12F_{2,2}\, F_{1,3}\, c^2 + 24F_{2,2}\, F_{2,1} - 24F_{2,2}\, F_{1,1} - 3F_{1,3}^2\, c^3 - 24F_{2,2}^2 - 24F_{1,2}\, F_{2,1} - 36F_{2,2}\, F_{1,3}\, c + 24F_{1,2}\, F_{1,1} + F_{1,2}\, F_{1,4}\, c^3 + 36F_{2,1}\, F_{1,3}\, c + 24F_{1,2}\, F_{2,2}$$

We show the following result:

Theorem: A curve $C(t) : t \in [a,b] \mapsto C(t) \in R^2$ with sufficient smoothness can be approximated by a quartic parametrization $t \mapsto P(t) \in R^2$ with order 7 at any point where $B^2 - 4AD \geq 0$.

Proof: The associated system is reduced to

$$e0 := x_{1,2} - c$$
$$e1 := 0 - x_{1,1}$$
$$e2 := F_{1,1} - y_{1,1}$$
$$e3 := F_{1,2}\, c - y_{1,2}$$
$$e4 := F_{1,3}\, c^2 + F_{1,2}\, x_{1,3} - y_{1,3}$$
$$e5 := F_{1,4}\, c^3 + 3F_{1,3}\, c x_{1,3} + F_{1,2}\, x_{1,4} - y_{1,4}$$

$$e6 := 1 - x_{2,1}$$
$$e7 := F_{2,1} - y_{2,1}$$
$$e8 := F_{2,2}\, x_{2,2} - y_{2,2}$$
$$e9 := F_{2,3}\, x^2_{2,2} + F_{2,2}\, x_{2,3} - y_{2,3}.$$

Since e9 is quadratic in $x_{2,2}$, we have either two real or two imaginary solutions. We apply the Maple program in Section 5 to do the calculations above to obtain the following solutions and conditions:

$$b1 = \frac{F_{1,2}\, c}{4} + F_{1,1},\quad a0 = 0,\quad b0 = F_{1,1},\quad a4 = 1,$$

$$b4 = F_{2,1},\quad a1 = \frac{c}{4},\quad a3 = R,\quad b3 = F_{2,2}\, R - F_{2,2} + F_{2,1}$$

$$a2 = \frac{-1}{36cF_{1,3}}\left(F_{1,4}\, c^3 - 12\, F_{1,3}\, c^2 + 24F_{1,2}\, R + 24\, F_{1,1} - 24\, F_{2,2}\, R + 24\, R_{2,2} - 24\, F_{2,1}\right),$$

$$b2 = \frac{-1}{36cF_{1,3}}\ ,(-3\, F_{1,3}^2\, c^3 + F_{1,2}\, F_{1,4}\, c^3 - 12\, F_{1,2}\, F_{1,3}\, c^2 + 24\, F_{1,2}^2\, R + 24\, F_{1,2}\, F_{1,1} -$$

$$24\, F_{1,2}\, F_{2,2}\, R + 24\, F_{1,2}\, F_{2,2} - 24\, F_{1,2}\, F_{2,1} - 36F_{1,1}\, F_{1,3}\, c\ )$$

where R is the root of

$$48\,F_{2,3}\,F_{1,3}\,cZ^2+\left(-96\,F_{1,3}\,F_{2,3}\,c+24\,F_{1,2}^2-48\,F_{1,2}\,F_{2,2}+24\,F_{2,2}^2\right)Z-$$
$$12\,F_{1,2}\,F_{1,3}\,c^2-36\,F_{1,1}\,F_{1,3}\,c+48\,F_{1,3}\,F_{2,3}\,c-F_{2,2}\,F_{1,4}\,c^3+12\,F_{2,2}\,F_{1,3}\,c^2+24\,F_{2,2}\,F_{2,1}-$$
$$24\,F_{2,2}\,F_{1,1}-3\,F_{1,3}^2\,c^3\,24\,F_{2,2}^2-24\,F_{1,2}\,F_{2,1}-36\,F_{2,2}\,F_{1,3}\,c+24\,F_{1,2}\,F_{1,1}+F_{1,2}\,F_{1,4}\,c^3+$$
$$36\,F_{2,1}\,F_{1,3}\,c+24\,F_{1,2}\,F_{2,2}+0.$$

Thus the existence of a solution requires that the above equation has real roots which confirms the theorem.

MAPLE-Program and Example

```
# First the quartic Bernstein polynomials are defined
bern : = proc(n,i,t)
binomial(n,I)*(1-t)^(n-1)*t^i
end:
# Input t1,t2: nodes of approximation, F[1,1], F[2,1]: value of the function at
# t1 and t2 and F[i,j], i= 1,2, j = 2,3,4: the associated Derivatives of the
# function to be approximated at t1, t2.
t1 : = 0; t2 : = 1;
# The quartic polynomials X(t) and Y(t) are defined by X and Y respectively
X : = a0 * bern(4,0,t) + a1 * bern(4,1,t) + a2 * bern(4,2,t) + a3 * bern(4,3,t) + a4 *
bern(4,4,t);
Y : = b0 * bern(4,0,t) + b1 * bern(4,1,t) + b2 * bern(4,2,t) + b3 * bern(4,3,t) + b4 *
bern(4,4,t);
# The derivatives of X and Y.
der X : = proc(j,t)
diff(X,t$j)
end:
der Y : = proc(j,t)
diff(Y,t$j)
end:
# We simplify the notations of derivatives of X and Y and substitute t = t1,t2.
X := array(1..2, 1..4);
y : = array(1..2, 1..4);
F : = array(1..2, 1..4);
for j from 2 to 4 do
x[1,j] : = subs(t = t1, der X (j - 1, t));
x[2,j] : = subs(t = t2, der X (j - 1, t));
od;
x[1,1] : = subs(t = t1, X); x[2,1] : = subs (t = t2,X);
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```
for j from 2 to 4 do
y[1,j] : = subs(t = t1, der Y (j - 1, t));
y[2,j] : = subs(t = t2, der Y (j - 1, t));
od;
y[1,1] : = subs(t = t1, Y); y[2,1] : = subs (t = t2, Y);
# Now we substitute in the system of equations
e0 : = x[1,2] - c;
e1 : = 0 - x[1,1];
e2 : = F[1,1] - y[1,1];
e3 : = F[1,2] * c - y[1,2];
e4 : = F[1,3] * c^2 + F[1,2] * x[1,3] - y[1.3];
e5 : = F[1,4] * c^3 + 3 * F[1,3] * c * x[1,3] + F[1,2] * x[1,4] - y[1,4];
e6 : = 1 - x[2,1];
e7 : = F[2,1] - y[2,1];
e8 : = F[2,2] * x[2,2] - y[2,2];
e9 : = F[2,3] * x[2,2]^2 + F[2,2] * x[2,3] - y[2,3];
sol : = solve({e1 = 0, e2 = 0, e3, 0, e3 = 0, e4 = 0, e5 = 0, e6 = 0, e7 = 0, e8 = 0, e9 = 0,
e0 = 0}
{a0, a1, a2, a3, a4, b0, b1, b2, b3, b4});
assign(sol);
```

We find a quartic approximant of *arcsinh*(t) by applying the Maple program with (t1,t2) = (-1,1). Modify the program by setting

$$t1 := -1,\ t2 := 1,\ c := 1,\ e1 := -1 - x[1,1]$$

and supply the program with the values of *arcsinh*(t) and its first, second, and third derivatives at t1 = -1

$$F[1,1] := -.8813735870,\ F[1,2] := .7071067810,\ F[1,3] := .3535533905,\ F[1,4] := .1767766953,$$

and the values of arcsinh (t) and its first, and second derivatives at t2 = 1 F[2,1] : = .8813735870, F[2,2] : = .7071067810, F[2,3] : = -.3535533905.

Executing the program gives two solutions

b0 = - .0724156341, a2 = .3851596407, a1 = .1525656862, a0 = - .0724922388,
b1 = .1520744030, b2 = .3889451777, a3 = .6502316994, b4 = .8813735870,
b3 = .6340500499, a4 = 1.

And

a3 = 1.349768312, b4 = .8813735870, b3 = 1.128697132, b0 = .1749079121,
a2 = 1.084696261, a0 = .277276081, b1 = .5230597193, a1 = .677218159,
a4 = 1., b2 = .8835922630.

And thus there are two curves $t \mapsto (X1(t), Y1(t))$ and $t \mapsto (X2(t), Y2(t))$, where

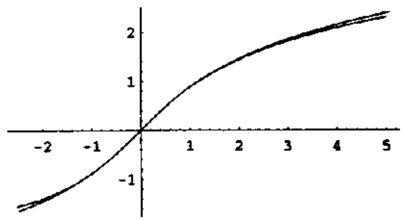
$$X1(t) := -0.0724922388\,(1-t)^4 + 0.6102627448\,(1-t)^3\,t + 2.310957844\,(1-t)^2\,t^2 + 2.600926798\,(1-t)\,t^3 + t^4,$$

$$Y1(t) := -\,0.0724156341\,(1-t)^4 + 0.6082976120\,(1-t)^3 + 2.333671066\,(1-t)^2\,t^2 + 2.536200200\,(1-t)\,t^3 + 0.8813735870t^4$$

$$X2(t) := -0.277276081\,(1-t)^4 + 2.708872636\,(1-t)^3\,t + 6.508177566\,(1-t)^2\,t^2 + 5.399073248\,(1-t)\,t^3 + t^4,$$

$$Y2(t) := -\,0.1749079121\,(1-t)^4 + 2.092238877\,(1-t)^3 + 5.301553578\,(1-t)^2\,t^2 + 4.514788528\,(1-t)\,t^3 + 0.8813735870t^4.$$

The figure shows that the curve $t \mapsto (X1(t), Y1(t))$ coincides graphically with *arcsinh(t)*, $\forall t \in [-2,5m5]$.



Approximation of arcsinh(t) in [-1,1]

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تقريب جزئي من الدرجة الرابعة باستخدام شكل بيرنشتاين

عبدالله ربابه
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الدوحة ، قطر

(قدم للنشر في ١٩/ ١١/ ١٤١٦هـ، وقبل للنشر في ١٧/ ٦/ ١٤١٧هـ)

ملخص البحث. ندرس ونعطي شروطًا ضرورية من الدرجة الرابعة لإيجاد تقريب بشكل بيرنشتاين للحصول على رتبة بدقّة عالية. نستعمل لغة برمجة رمزية (مابل) لكتابة برنامج لتسهيل الطريقة وجعلها سهلة في التطبيقات. أوضحت الطريقة إمكانيّة الحصول على رتبة تقريب من الدرجة السابعة باستخدام تقريب جزئي من الدرجة الرابعة على أساس بيرنشتاين.