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Influence of Shear Deformation and Rotary Inertia on Nonlinear Free Vibration of a Beam with Pinned Ends

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Abstract. The method of multiple scales is used to analyze the nonlinear vibrations of a beam with pinned ends. The formulation incorporates the effects of the transverse shear deformation as well as the rotary inertia on the large-amplitude vibration behaviour. A uniformly valid second-order perturbation solution is obtained. Predictions of the nonlinear frequencies for different beam parameters are given. The influences of shear and rotary inertia are significant for moderately thick and short beams undergoing large amplitude vibrations.

Introduction

During the past few years, the nonlinear vibrations of classical Bernoulli-Euler beams undergoing large deflections have been extensively investigated by many authors [1-12]. With the intensive use of composite materials having relatively high transverse shear moduli and the need for beam members with high natural frequencies in areospace, civil and mechanical engineering applications, a more refined higher-order theory is called for. Consequently, the Timoshenko beam model, which includes shear deformation and rotary inertia effects, must be used in the investigation of the dynamic response and stability problems of beams. The Timoshenko beam theory has an extended range of applications because it allows treatment of deep beams (depth is large relative to length), short and thin-webbed beams and beams where higher modes are excited. However, it introduces some complications not found in the elementry Bernoulli-Euler formulation.

Although there is an ample literature on the large-amplitude vibrations of Bernoulli-Euler beams, only few publications were devoted to include the effects of shear deformation and rotary inertia. The inclusion of the aforementioned effects as well as compressive axial loads on the linear vibrations of beams has been studied by several authors. Lunden and Akesson [13] used complex stiffness matrix method to investigate the damped linear vibration of Rayleigh-Timoshenko beam subjected to a harmonic nonresonant loading. Chandrashekhara *et al.* [14] presented exact solutions for linear free vibration of symmetrically laminated composite beam including shear deformation and rotary inertia for some arbitrary boundary conditions. Abramovich [15] investigated the influence of compressive axial loads on the linear frequencies of Timoshenko type beams having various boundary conditions. That formulation neglected the joint action of rotary inertia and shear deformation effect. Following the same approach, Abramovich [16] studied the effects of shear deformation and rotary inertia on linear vibration of symmetrically laminated composite beams. A very recent work by Hou *et al.* [17] used a finite element model, with shear deformation and rotary inertia taken into account, to investigate the small amplitude free vibration of nonuniform beams resting on varying elastic foundations.

Rao *et. al.* [18] linearized the nonlinear strain-displacement relations and employed the standard principles that are used in finite element analyses to formulate the linearized stiffness and mass matrices. The resulting linear algebraic eigenvalue problem is solved by employing an iterative technique that was used previously by Chuh [19]. Using a finite-element formulation, Shastry and Rao [20] investigated the dynamic stability of bars subjected to axial forces, including shear-deformation and rotary inertia effects.

In this paper, the large amplitude free vibration of a simply supported Timoshenko beam is considered. The solution of the governing equations is obtained by utilizing a space averaging tecnnique. The method of multiple scales is then used to determine a second-order perturbation solution. The results presented show a coupling effect between the two bands of frequencies and give basic understanding of the influence of shear deformation and rotary inertia on the nonlinear frequencies.

Basic Equations

Consider a beam of uniform cross-section made of homogenous isotropic material without damping pinned at its immovable ends. When the amplitude of vibration is of the same order of the depth of the beam, elementary linear beam theory must be extended to include the nonlinear stretching force due to large displacement of the middle plane of the beam. In this paper, we use the strain-displacement relations of the von Karman type

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 , \quad k_x = -\frac{\partial^2 w}{\partial x^2} \tag{1}$$

where ε_x and k_x are the mid-plane strain and curvature respectively, u and w are the the axial and transverse displacements, respectively, and x is the axial coordinate of the beam.

Two corrections are used to refine the elementary Bernoulli-Euler theory. The first is the contribution of rotational energy of the beam cross-section due to the rotary inertia, which becomes increasingly important for shorter bending wavelengths. Such a correction was already considered by Lord Rayleigh (Rayleigh beam) [21]. The second correction is the effect of shear deformation which is caused by shear stresses acting on the beam cross-section. Timoshenko [22] was the first to include the second correction, which is usually more significant than the first when the bending wavelength is small compared to the cross sectional dimensions of the beam. Shear deformations contribute to the change in slope of the beam. Therefore the slope may be written as

$$\frac{\partial w}{\partial x} = \psi + \gamma , \tag{2}$$

where ψ is the rotation due to bending and γ represents the shear angle. The shear force V is given by

$$V = -k A G \gamma , \tag{3}$$

where A is the cross-sectional area of the beam, G is the shear modulus which equals E/2 $(1+\nu)$, with ν being Poisson's ratio. E is Young's modulus; and k is the shear-correction factor which depends on the shape of cross section as discussed by Cowper [23]. For I-beam cross sections, the shear correction factor is 0.4167, or even smaller. For rectangular cross sections, this factor is 0.8333, and for circular sections it is 0.8475.

The kinetic energy T and the strain energy U of the beam can be written as

$$T = \frac{1}{2}\int_{o}^{L}\left[m\left(\frac{\partial w}{\partial t}\right)^2 + m\left(\frac{\partial u}{\partial t}\right)^2 + m r^2\left(\frac{\partial \psi}{\partial t}\right)^2 \right] dx , \tag{4}$$

$$U = \frac{1}{2}\int_{o}^{L}\left[EI\left(\frac{\partial \psi}{\partial x}\right)^2 + k A G \gamma^2 + E A \varepsilon^2 \right] dx , \tag{5}$$

where I is the second moment of area of the cross-section, L is the length of the beam, m is the mass per unit length, and r is the radius of gyration ($r^2 = I/A$). According to Hamilton's principle, variation of the Lagrangian, $\int_{t_o}^{t} \delta[T - U] = 0,$ where t is the time, provide the governing equations and boundary conditions. After some manipulation, the governing equations can be written as

$$\frac{\partial}{\partial x}\left(EI \frac{\partial w}{\partial x}\right) + k A G\left(\frac{\partial w}{\partial x} - \psi\right) - m r^2 \frac{\partial^2 \psi}{\partial t^2} = 0 \tag{6}$$

$$m\frac{\partial^2 w}{\partial t^2}-\frac{\partial}{\partial x}\left[kAG\left(\frac{\partial w}{\partial x}-\psi\right)\right]-\frac{\partial}{\partial x}\left\{EA\left[\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\frac{\partial w}{\partial x}\right\}=0, \tag{7}$$

$$m\frac{\partial^2 u}{\partial t^2}-\frac{\partial}{\partial x}\left\{EA\left[\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\right\}=0 \tag{8}$$

The boundary conditions for a simply supported beam with immovable ends are

$$u=w=0,\ \frac{\partial\psi}{\partial x}=0 \text{ at } x=0 \text{ and } x=L \tag{9}$$

The nonlinearity is caused by the pinned ends not being allowed to move relative to the initial coordinates of the beam ends. Equations (6)-(8) are coupled and nonlinear. Due to the complexity of these equations, the only present means of solution are approximate methods.

Upon neglecting the axial inertia, Eq.(8) can be integrated with respect to x and the result is substituted into Eq.(7) to arrive at the following set of coupled differential equations for beams with constant properties:

$$EI\frac{\partial^2 w}{\partial x^2}+kAG\left(\frac{\partial w}{\partial x}-\psi\right)-mr^2\frac{\partial^2 \psi}{\partial t^2}=0 \tag{10}$$

$$m\frac{\partial^2 w}{\partial t^2}-kAG\left(\frac{\partial^2 w}{\partial x^2}-\frac{\partial\psi}{\partial x}\right)-N_x\frac{\partial^2 w}{\partial x^2}=0, \tag{11}$$

where the axial force N_x is given by

$$N_x=N_0+\frac{EA}{2L}\int_0^L\left(\frac{\partial w}{\partial x}\right)^2 dx, \tag{12}$$

and N_0 is the initial axial tensile force if it exists.

Equations (10) and (11), with the last nonlinear term neglected in the later, are usually referred to as the Timoshenko beam equations. One can decouple Eqs.(10) and (11) to arrive at the following equation governing the transverse deflection w:

$$EI\frac{\partial^4 w}{\partial x^4}+m\frac{\partial^2 w}{\partial t^2}-\left(mr^2+\frac{mEI}{kAG}\right)\frac{\partial^4 w}{\partial x^2\partial t^2}+\frac{m^2 r^2}{kAG}\frac{\partial^4 w}{\partial t^4}$$
$$-N_x\left[\frac{\partial^2 w}{\partial x^2}-\frac{EI}{kAG}\frac{\partial^4 w}{\partial x^4}+\frac{mr^2}{kAG}\frac{\partial^4 w}{\partial x^2\partial t^2}\right]=0 \qquad (13)$$

It may be noted that the equation governing the rotation of the cross-section ψ is the same as Eq.(13) with w being repalced by ψ.

The first two terms in Eq.(13) correspond to the classical Bernoulli- Euler beam model. The third term represents the correction for rotary inertia while the fourth term represents the shear deformation effect. The fifth term represents the joint action of rotary inertia and shear deformation. The last bracketed term represents the effect of the axial force N_x, caused by the nonlinearity due to large amplitudes.

Sato [24] used the extended Hamilton principle to derive a similar equation for a Timoshenko beam with a constant external force following the deflection of the beam at its tip. Equations (10) and (11) are more accurate than the equations derived from the dynamic equilibrium conditions of a beam element by Kolousek [25] and have been applied to vibration and stability analyses of Timoshenko column undergoing tangential follower force by Kounadis [26].

It is convenient to work with dimensionless quantities, because such a formulation will facilitate the identification of the order of magnitude of some variables. Hence, let

$$\hat{t}=\omega t,\ \hat{x}=x/L,\ \hat{w}=w/L \qquad (14)$$

where $\omega=(n^2\pi^2/L^2)\sqrt{EI/m}$ is the linear circular frequency for a Bernoulli-Euler beam. Substituting Eq.(14) into Eq.(13) and using the chain rule of differentiation, we put Eq.(13) in the nondimensional form

$$\frac{\partial^4 w}{\partial x^4}+\frac{m\omega^2 L^4}{EI}\frac{\partial^2 w}{\partial t^2}-\frac{mt^2\omega^2 L^2}{EI}\left(1+\frac{E}{kG}\right)\frac{\partial^4 w}{\partial x^2\partial t^2}+\frac{m^2 r^2\omega^2 L^4}{kAGEI}\frac{\partial^4 w}{\partial t^4}$$
$$-N_x\left[\frac{\partial^2 w}{\partial x^2}-\frac{EI}{kAGL^2}\frac{\partial^4 w}{\partial x^4}+\frac{m\omega^2 r^2}{kAG}\frac{\partial^4 w}{\partial x^2\partial t^2}\right]=0, \qquad (15)$$

where

$$N_x=\frac{N_o L^2}{EI}+\frac{AL^2}{2I}\int_0^1\left(\frac{\partial w}{\partial x}\right)^2 dx \qquad (16)$$

We note that in the above and hereinafter we drop the caret on all the quantities and variables for notational convenience.

In dimensionless form, the boundary conditions, Eqs.(9), become

$$w(x,t) = \frac{\partial^2 \psi}{\partial x^2} = 0 \text{ at } x = 0 \text{ and } x = 1 \tag{17}$$

To proceed with the analysis of Eq.(15), we reduce it to an ordinary diferential equation by utilizing the averaging technique over the space variable (Galerkin method) to yield an alternate form amenable to solution. To this end, we let

$$w(x,t) = \phi(x)\, q(t) \tag{18}$$

where f(x) is the charcteristic mode of a simply supported beam; that is,

$$\phi(x) = \sin(n\pi x), \quad n=1,2,3,.... \tag{19}$$

At this point, it will prove convenient to introduce the following nondimensional parameters:

$$\lambda = E/kG, \quad \eta = (L/n\pi r)^2 \tag{20}$$

Substituting Eqs.(18) and (19) into Eq.(15) and performing the integration in Eq.(16) one obtains the following nonlinear fourth order ordinary differential equation after a bit of handling:

$$\ddddot{q}+(\alpha_1+\alpha_2 q^2)\ddot{q}+\alpha_3 q+\alpha_4 q^3=0, \tag{21}$$

where the over dot signifies the derivative with respect to time and the coefficients α_i are given as

$$\alpha_1 = \eta(1+1/\lambda+\eta/\lambda+N_o/EA), \tag{22}$$

$$\alpha_2 = n^2\pi^2\eta/4, \tag{23}$$

$$\alpha_3 = \eta^2[1/\lambda+(N_o/EA)(1+\eta/\lambda)], \tag{24}$$

$$\alpha_4 = n^2\pi^2\eta^2(1+\eta/\lambda)/4. \tag{25}$$

Equation (21) represents a single degree-of-freedom vibratory system with inertia and stiffness nonlinearities. Since Eq.(21) is of fouth order, one needs to specify four initial conditions. In this paper, we let

$$w(1/2,0) = w_{max}/L, \quad \dot{w}(1/2,0) = \ddot{w}(1/2,0) = \dddot{w}(1/2,0) = 0 \tag{26a}$$

where w_{max} is the central amplitude. This poses the following initial conditions on q:

$$q(0) = w_{max} / L,\ \dot{q}(0) = \ddot{q}(0) = \dddot{q}(0) = 0 \tag{26b}$$

Application of the Method of Multiple Scales

A number of techniques are applicable in seeking approximate solution to Eq.(21). These include the method of harmonic balance, Lindstedt-Poincare', equivalent linearization, Krylov-Bogliubv-Mitropolski, generalized averaging, and the multipe scales method. In the former three methods, one seeks a periodic solution which is assumed *a priori*. The latter three methods yield a set of first order differential equations, which describe the slow time evolution of the amplitude and phase of the response. These methods have been described throughly by Nayfeh [27, 28] and Nayfeh and Mook [29]. Our approach in seeking an analytical solution to Eq. (21) is to use the method of multiple scales.

To this end, we seek a second-order uniform expansion of the solution of Eq.(21) in the form

$$q(t) = \sum_{j=1}^{3} \varepsilon^j q_j(T_o, T_1, T_2) + \ldots., \tag{27}$$

where ε is a small parameter that measures the amplitude of oscillation. It is used as a book-keeping device and set equal to unity if the amplitude is taken to be small. One defines a fast time scale $T_0 = t$, on which the main oscillatory behaviour occurs, and slow time scales $T_j = \varepsilon^j t$, $j > 1$, on which the amplitude and phase modulation takes place. The time derivatives are thus expanded as

$$\frac{d}{dt} = D_o + \varepsilon D_1 + \varepsilon^3 D_2 + \ldots., \tag{28}$$

$$\frac{d^2}{dt^2} = D_o^2 + 2\varepsilon D_o D_1 + \varepsilon^2 (D_1^2 + 2 D_o D_2) + \ldots., \tag{29}$$

$$\frac{d^4}{dt^4} = D_o^4 + 4\varepsilon D_o^3 D_1 + \varepsilon^2 (6 D_o^2 D_1^2 + 4 D_o^3 D_2) + \ldots., \tag{30}$$

where

$$D_j = \frac{\partial}{\partial T_j}$$

One next substitutes Eqs.(27) and (28)-(30) into Eq.(21) and equate the like powers of ε to zero to obtain the following hierarchy of linear differential equations which are to be solved successively. These equations are

$$D_o^4 q_1 - \alpha_1 D_o^2 q_1 + \alpha_3 q_1 = 0, \tag{31}$$

$$D_o^4 q_2 + \alpha_1 D_o^2 q_2 + \alpha_3 q_2 = -4 D_o^3 D_1 q_1 - 2\alpha_1 D_o D_1 q_1 \tag{32}$$

$$D_o^4 q_3 + \alpha_1 D_o^2 q_3 + \alpha_3 q_3 = -(6 D_o^2 D_1^2 + 4 D_o^3 D_2) q_1 - \alpha_1 (D_1^2 + 2 D_o D_2) q_1$$
$$- \alpha_2 q_1^2 D_o^2 q_1 - 4 D_o^3 D_1 q_2 - 2\alpha_1 D_o D_1 q_2 - \alpha_4 q_1^3 \tag{33}$$

The general solution of Eq.(31) is expressed in the following complex form

$$q_1(T_o, T_1, T_2) = A_1(T_1, T_2) e^{i\omega_1 T} + A_2(T_1, T_2) e^{i\omega_2 T} + cc, \tag{34}$$

where cc stands for the complex conjugate of the preceding terms and the notation indicates that only the scales T_0, T_1, and T_2 are used because the the solution is to be obtained to second order. The complex amplitudes $A_j(T_1,T_2) = {}^1/_2\, a_j(T_1, T_2) \exp[i\, \beta_j(T_1,T_2)]$ are functions of the slow scales with both the amplitude a_j and phase β_j being real. The linear frequencies $\omega_{1,2}$ are given by

$$\omega_{1,2}^2 = \frac{\alpha_1}{2} + \sqrt{\frac{\alpha_1^2}{4} - \alpha_3}\,. \tag{35}$$

Equation.(35) indicates that, for a Timoshenko type beam, there are two bands of linear frequencies denoted by the subscripts 1 (lower) and 2 (higher), respectively. In other words there are two sinusoidal modes of different frequencies corresponding to the same spatial mode. This is the most important difference between Timoshenko-type and Bernoulli-Euler beams, or a beam in which the correction due to shear or rotary inertia is neglected. The orders of magnitude of these frequencies were discussed by Abramovich and Elishakoff [30] and their existence has been demonstrated experimentally by Barr (see ref. 30). Recently, Gopalakrishnan *et al.* [31] used a spectral analysis to explore the presence of the two propagating wave modes that correspond to the fundamental frequencies in these bands.

The use of Eq.(34) in Eq.(32) then yields the following inhomogeneous equation for $q_2(T_0, T_1, T_2)$:

$$D_o^4 q_2 + \alpha_1 D_o^2 q_2 + \alpha_3 q_2 = 2i\omega_1 (2\omega_1^2 - \alpha_1) D_1 A_1 e^{i\omega_1 T_o}$$
$$+ 2i\omega_2 (2\omega_2^2 - \alpha_1) D_1 A_2 e^{i\omega_2 T_o} + cc, \tag{36}$$

The right-hand side of Eq.(35) is resonant and one requires it vanishes in order to avoid secular behaviour in q_2. This is accomplished by setting

$$D_1 A_1 = 0, \quad D_1 A_2 = 0 \tag{37a}$$

or equivalently

$$A_1 = A_1(T_2), \quad A_2 = A_2(T_2). \tag{37b}$$

These equations define the rate of changes of the amplitudes A_1 and A_2 on the slow time scale T_1. Consequently, q_2 is governed by a homogeneous differential equation. According to the method of multiple scales one does not need the homogeneous solution for q_2. Hence q_2 is set equal to zero.

Substituting Eq.(34) into Eq.(33) and recalling the fact that $q_2=0$ yields

$$\begin{aligned} D_o^4 q_3 + \alpha_1 D_o^2 q_3 + \alpha_3 q_3 = & \{2i\omega_1(2\omega_1^2-\alpha_1)D_2A_1 + 3(\alpha_2\omega_1^2-\alpha_4)A_1^2\overline{A}_1 \\ & +2[\alpha_2(\omega_1^2+2\omega_2^2)-3\alpha_4]A_1A_2\overline{A}_2\}e^{i\omega_1 T_o} \\ & +\{2i\omega_2(2\omega_2^2-\alpha_1)D_2A_2+3(\alpha_2\omega_2^2-\alpha_4)A_2^2\overline{A}_2 \\ & +2[\alpha_2(\omega_2^2+2\omega_1^2)-3\alpha_4]A_1\overline{A}_1A_2\}e^{i\omega_2 T_o} \\ & +(\alpha_2\omega_1^2-\alpha_4)A_1^3e^{3i\omega_1 T_o}+(\alpha_2\omega_2^2-\alpha_4)A_2^3e^{3i\omega_2 T_o} \\ & +[\alpha_2(\omega_1^2+2\omega_2^2)-3\alpha_4]\{A_1A_2^2e^{i(\omega_1+2\omega_2)T_o}+A_1\overline{A}_2^2e^{i(\omega_1-2\omega_2)T_o}\} \\ & +[\alpha_2(\omega_2^2+2\omega_1^2)-3\alpha_4]\{A_1^2A_2e^{i(2\omega_1+\omega_2)T_o}+A_1^2\overline{A}_2e^{i(2\omega_1-\omega_2)T_o}\} \\ & +cc, \end{aligned} \qquad (38)$$

where $\overline{A}_j$ is the complex conjugate of A_j. The annulment of secular terms in q_3 requires that

$$2i\omega_1(2\omega_1^2-\alpha_1)D_2A_1+3(\alpha_2\omega_1^2-\alpha_4)A_1^2\overline{A}_1+2[\alpha_2(\omega_1^2+2\omega_2^2)-3\alpha_4]A_1A_2\overline{A}_2=-0. \qquad (39)$$

$$2i\omega_2(2\omega_2^2-\alpha_1)D_2A_2+3(\alpha_2\omega_2^2-\alpha_4)A_2^2\overline{A}_2+2[\alpha_2(\omega_2^2+2\omega_1^2)-3\alpha_4]A_1A_2\overline{A}_2=-0. \qquad (40)$$

We express A_j ($j = 1,2$) in the polar forms

$$A_1=\frac{1}{2}a_1e^{i\beta_1}$$
$$A_2=\frac{1}{2}a_2e^{i\beta_2} \qquad (41)$$

where a_j and β_j are real.

Substituting Eqs. (41) into Eqs.(39) and (40) and separating real and imaginary parts in each and then solving the resulting differential equations, we find that a_1 and a_2 are constant and hence that

$$\begin{aligned}\beta_1 &= \delta_1 T_2 + \beta_{10}\\ \beta_2 &= \delta_2 T_2 + \beta_{20}\end{aligned} \tag{42}$$

where β_{10} and β_{20} are constant and

$$\begin{aligned}\delta_1 &= \frac{3(\alpha_2\omega_1^2-\alpha_4)a_1^2+2[\alpha_2(\omega_1^2+2\omega_2^2)-3\alpha_4]a_2^2}{8\omega_1(2\omega_1^2-\alpha_1)},\\ \delta_2 &= \frac{3(\alpha_2\omega_2^2-\alpha_4)a_2^2+2[\alpha_2(\omega_2^2+2\omega_1^2)-3\alpha_4]a_1^2}{8\omega_2(2\omega_2^2-\alpha_1)},\end{aligned} \tag{43}$$

Returning to Eq. (41), one finds

$$\begin{aligned}A_1 &= \frac{1}{2}a_1\exp[i\varepsilon^2\delta_1 t + i\beta_{10}],\\ A_2 &= \frac{1}{2}a_2\exp[i\varepsilon^2\delta_2 t + i\beta_{20}],\end{aligned} \tag{44}$$

where $T_2 = \varepsilon^2 t$ is used.

Next, we solve for q_3, retaining only the particular solution, and then combine the resulting expression with q_1 from Eq. (34) according to Eq. (27), note that $q_2 = 0$, and arrive at the following uniformly valid expression for q in real functional form, after accounting for the complex conjugate.

$$\begin{aligned}q = {} & \varepsilon[a_1\cos(\Omega_1 t+\beta_{10})+a_2\cos(\Omega_2 t+\beta_{20})]\\ & +\varepsilon^3\{C_1 a_1^3\cos(3\Omega_1 t+3\beta_{10})+C_2 a_2^3\cos(3\Omega_2 t+3\beta_{20})\\ & +C_3 a_1 a_2^2\cos[(\Omega_1+2\Omega_2)t+\beta_{10}+2\beta_{20}]\\ & +C_4 a_1 a_2^2\cos[(\Omega_1+2\Omega_2)t+\beta_{10}+2\beta_{20}]\\ & +C_5 a_1^2 a_2\cos[(2\Omega_1+\Omega_2)t+2\beta_{10}+2\beta_{20}]\\ & +C_6 a_1^2 a_2\cos[(2\Omega_1+\Omega_2)t+\beta_{10}+2\beta_{20}]\end{aligned} \tag{45}$$

where the nonlinear frequencies Ω_j are given by

$$\Omega_j = \omega_j + \varepsilon^2\delta_j + o(\varepsilon^3),\ j = 1,2, \tag{46}$$

and the coefficients C_j, j=1 to 6, are

$$C_1 = \frac{\alpha_2 \omega_1^2 - \alpha_4}{4(81\omega_1^4 - 9\alpha_1\omega_1^2 + \alpha_3)},$$
$$C_2 = \frac{\alpha_2 \omega_2^2 - \alpha_4}{4(81\omega_2^4 - 9\alpha_1\omega_2^2 + \alpha_3)},$$
$$C_3 = \frac{\alpha_2 (\omega_1^2 + 2\omega_2^2) - 3\alpha_4}{4[(\omega_1 + 2\omega_2)^4 - \alpha_1(\omega_1 + 2\omega_2)^2 + \alpha_3]}$$
$$C_4 = \frac{\alpha_2 (\omega_1^2 + 2\omega_2^2) - 3\alpha_4}{4[(\omega_1 - 2\omega_2)^4 - \alpha_1(\omega_1 - 2\omega_2)^2 + \alpha_3]}$$
$$C_5 = \frac{\alpha_2 (\omega_1^2 + 2\omega_2^2) - 3\alpha_4}{4[(\omega_1 + 2\omega_2)^4 - \alpha_1(\omega_1 + 2\omega_2)^2 + \alpha_3]}$$
$$C_6 = \frac{\alpha_2 (\omega_1^2 - 2\omega_2^2) - 3\alpha_4}{4[(\omega_1 - 2\omega_2)^4 - \alpha_1(\omega_1 - 2\omega_2)^2 + \alpha_3]} \qquad (47)$$

The satisfaction of the initial conditions, Eqs. (26), requires that

$$a_1 = \frac{\omega_2^2 \, w_{max}}{(\omega_2^2 - \omega_1^2) L}$$
$$a_2 = \frac{\omega_1^2 \, w_{max}}{(\omega_1^2 - \omega_2^2) L} \qquad (48)$$
$$\beta_{10} = \beta_{20} = 0.$$

It is to be pointed out that the small parameter ε, which has now served its purpose, may be set equal to 1.

We note that the Bernoulli-Euler beam nondimensional nonlinear frequency can be recovered by setting, separately, $mr^2 = 0$ (i.e., rotational inertia is zero) and $\lambda=0$ (no shear deformation) in the expression for Ω_1 given by Eq. (46). This gives

$$\Omega_B = 1 + \frac{3}{8}\left(\frac{w_{max}}{2r}\right)^2 \qquad (49)$$

which is the same as that given by previous investigators, (see Refs. [6,7,8]). Also the nonlinear frequency for a Rayleigh beam can be recoverd if one sets $\lambda = 0$ (i. e., shear stiffness is infinite). The result is

$$\Omega_R = [1+\frac{3}{8}\left(\frac{w_{max}}{2r}\right)^2](1+n^2\pi^2 r^2/L^2)^{1/2} \tag{50}$$

For a beam in which the shear is taken into account but the rotary inertia is neglected (shear beam), the nonlinear frequency is as follows:

$$\Omega_s = \left[1+\frac{3}{8}\left(\frac{w_{max}}{2r}\right)^2\left(1+\frac{EIn^2\pi^2}{kGAL^2}\right)\right]\Big/\left(1+\frac{EIn^2\pi^2}{kGAL^2}\right)^{1/2} \tag{51}$$

Numerical Example

As an example, a simply supported beam made of aluminum undergoing flexural vibrations is considered. The assumed properties are taken from Ref. 31 and are given in Table 1. The cross-section is 25mm x 6mm.

Table 1. Material properties

Material property	Value
E	70 GPa
G	27 GPa
ρ	2700 Kg/m^3
k	0.85

The ratio of the nonlinear period of vibration, T, to the linear period, T_0, of the classical beam theory has been computed for different slenderness ratios and central amplitude-depth ratios. The results are presented in graphical forms. In all of the figures, we consider the responses that correspond to the first spatial mode of vibration (n=1). In addition, the nonlinear period T for a Timoshenko-type beam corresponds to the lower fundamental frequency Ω_1 ($T/T_0 = 1/\Omega_1$).

In Fig.1, the slenderness ratio is L/r=10, which corresponds to L/h=2.8867, where h is the depth of the beam. The nondimensional linear frequencies for a Timoshenko beam are ω_1=0.8518 and ω_2=6.8107. The figure shows that the influences of shear deformation and rotary inertia are paramount for thick beams. This means that the nonlinear natural frequency, which was predicted from classical theory deviates considerably from the actual one. Comparing the responses of a shear beam and a Rayleigh beam, one concludes that the effect of shear deformation is more significant than the effect of rotary inertia on the

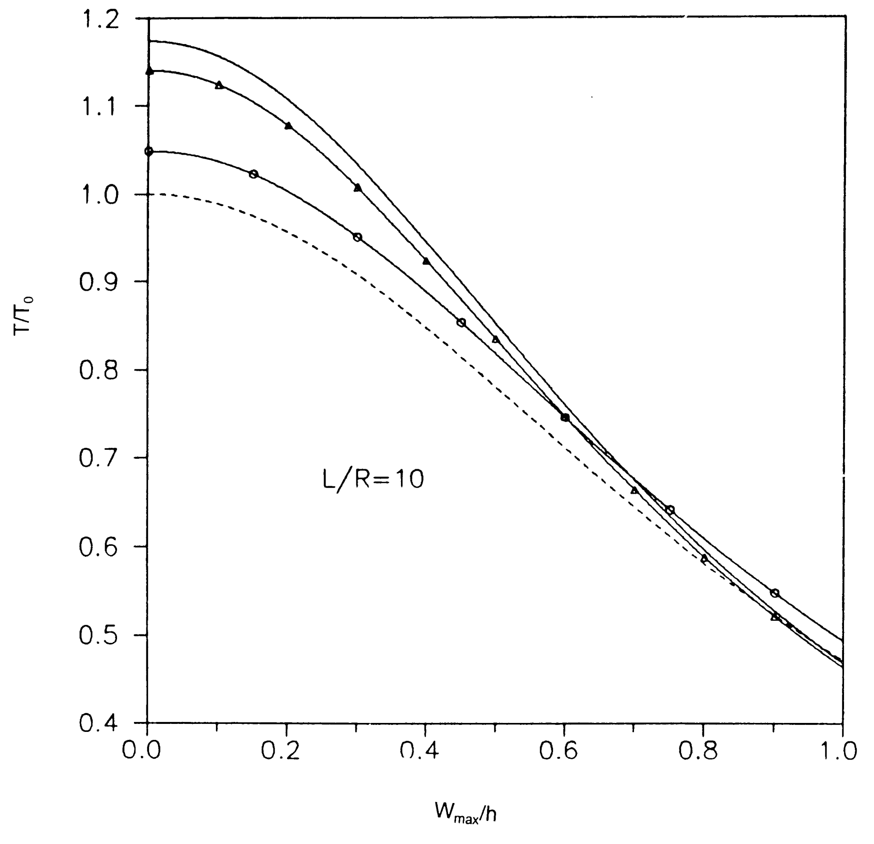

Fig. 1. Period of free vibration of a simply supported beam for L/r=10. (----------): Timoshenko beam, (-Δ-Δ-Δ-): shear beam, (o-o-o-o-): Rayleigh beam, (------------): Bernoulli beam.

large-amplitude behaviour of beams. However, as the central amplitude-depth ratio increases, the effect of rotary inertia bcomes more pronounced.

In Fig. 2, the slenderness ratio is L/r=20, which corresponds to L/h=5.7735. The values of ω_1 and ω_2 are 0.9542 and 24.319, respectively. The effects of shear deformation and rotary inertia are significant for small central amplitude-depth ratio. The behaviours of slender beams are depicted in Fig. 3. For slenderness ratio L/r=50 (L/h=14.4338), the linear frequencies are ω_1 = 0.998 and ω_2 = 581.3. From the figure it is difficult to differentiate between the responses of the different beams. Hence, as expected, the influences of shear deformation and rotary inertia can be neglected.

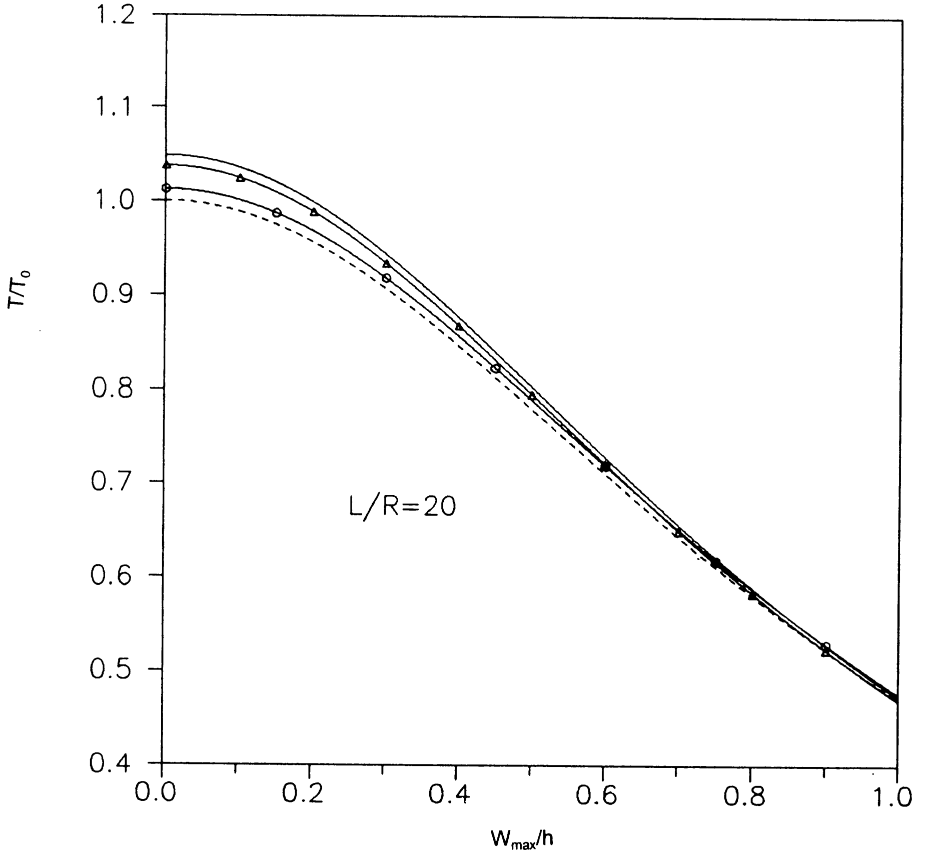

Fig. 2. Period of free vibration of a simply supported beam for L/r=20. (----------): Timoshenko beam, (-Δ-Δ-Δ-): shear beam, (o-o-o-o-): Rayleigh beam, (-------------): Bernoulli beam.

Conclusion

The secondary effects of transverse shear deformation and rotary inertia on the large-amplitude vibration at the first spatial mode of a pinned-end beams are studied. The method of multiple scales is used to find a second-order perturbation solution. These secondary effects may considerably affect the response of short and thick beams. Therefore, when the theory of beams is employed as a basis for the study of complicated structures,

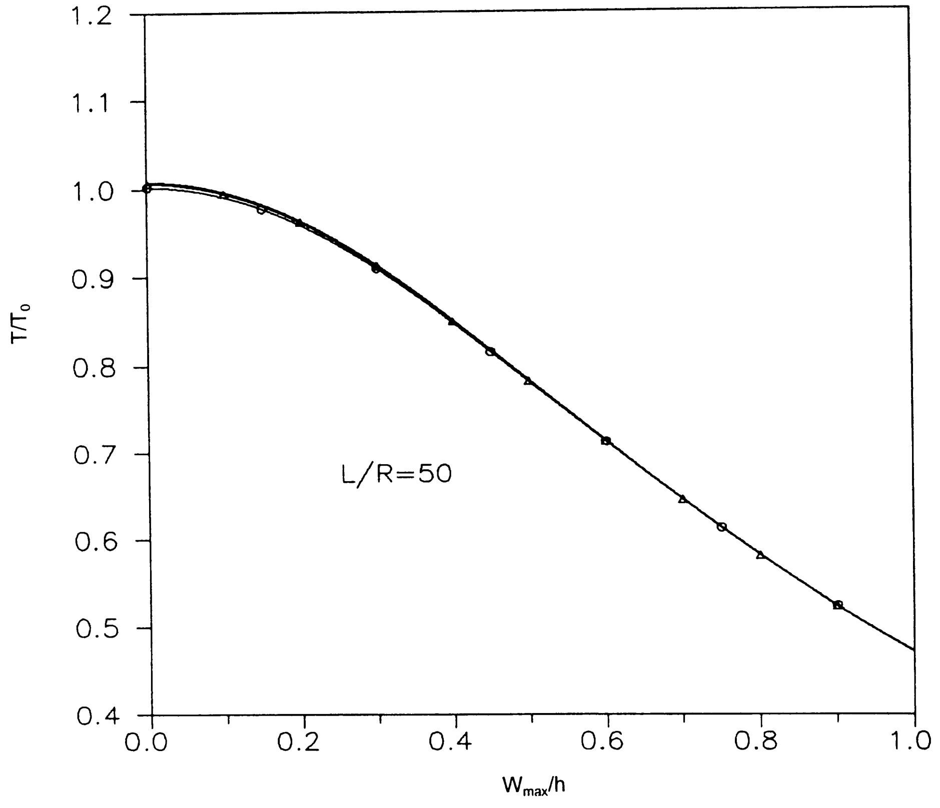

Fig. 3. Period of free vibration of a simply supported beam for L/r=50. (---------): Timoshenko beam, (-Δ-Δ-Δ-): shear beam, (o-o-o-o-): Rayleigh beam, (-----------): Bernoulli beam.

such as wings of airplanes and missiles structures, it is imperative that Timoshenko beam theory be employed in the modeling. Consequently, the shear and rotrary inertia effects are captured for an accurate dynamic analysis.

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تأثير تشوهات القص وعزم القصور الدوراني على الاهتزازات الحُرّة اللاخطّية لكمرة مثبّتة تثبيتاً بسيطاً

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(أستلم في ٢٣/٦/١٩٩٦م ؛ وقُبل للنشر في ١٤/١٢/١٩٩٦م)

ملخص البحث. لقد استخدمت طريقة المقاييس متعددة الأضطراب لفحص الاهتزازات اللاخطية لكمرة مثبّتة تثبيتاً بسيطاً. تضمنت المعادلات وحلولها تأثيرات تشوهات القص العرضية وعزم القصور الدوراني على الاهتزازات العرضية ذات السعة الكبيرة. كانت هذه التأثيرات ملموسة بالنسبة للكمرات السميكة والقصيرة.

لقد تم توصيف الإزاحات العرضية بمسلسلة منتظمة صحيحة من الدرجة الثانية. وتم إيجاد الزمن الدوري اللاخطي عند قيم مختلفة للمتغيرات الموصّفة للكمرة.