

Systems with Exponential Life and NBAFR Component Lives

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Abstract. For systems with exponential life formed from independent NBAFR components it is shown that (i) if the system is monotone then the system must be a series system of exponential components and (ii) if the system life is the sum of component lives then $(n-1)$ components are degenerate at zero and the remaining one is exponential.

Introduction

A nonnegative random variable X having distribution function $F(x)$ has an increasing failure rate on the average (IFRA) if $-(1/t)\log\bar{F}(t)$ is nondecreasing in $t \geq 0$ where $\bar{F}(t) = 1 - F(t)$. The IFRA class of probability distributions is of a great importance in reliability theory because of the IFRA closure theorem [1]. Recently, Loh [2] introduced the new better than used average failure rate (NBAFR) class of distributions. This class is defined as: F is NBAFR if and only if $L = \int_0^{\infty} \frac{(-\log\bar{F}(x))}{x} dx$ exists (possibly infinite) and $-\log\bar{F}(x) \geq Lx$. The NBAFR does not only contain the IFRA distributions but also the new better than used (NBU) class of life distributions (The random variable X has a NBU distribution if $\bar{F}(x+y) \leq \bar{F}(x)\bar{F}(y)$). For some closure properties of the NBAFR class one may consult Loh [2] and Alzaid *et al.* [3].

Remark 1

In his definition for the NBAFR distributions Loh [2] assumed that $F(0) = 0$ which excludes those distributions with mass at zero. This assumption will not be made here since we need to include distributions which have a probability mass at zero. A similar definition is used by Barlow and Proschan [1, p. 87] when considering the IFRA class of distributions.

Block and Savits [4] studied systems with exponential life formed from independent IFRA components. They showed that in the case of monotonic systems [1, p.

84], the system must be essentially (except for degenerate) a series system of exponential components. They also proved that in the case of systems whose life is the sum of component lives, all but one of the components are degenerate at zero while the remaining one is exponential. Shaked [5] improved their result by requiring NBU components instead of IFRA components. In this paper it is shown that these results still hold under the weaker assumption that the components are NBAFR.

Main Result

Theorem 1

Let $F(t)$ be the life distribution of a monotonic system of order n formed of independent NBAFR components having life distributions $F_1(t), \dots, F_n(t)$ respectively and assume $F(t)$ is exponential. Then there exists a subcollection $1 \leq i_1 < \dots < i_k \leq n$ such that for all $t \geq 0$

$$(1) \quad \hat{F}(t) = \hat{F}_{i_1}(t) \dots \hat{F}_{i_k}(t)$$

where each $F_{i_j}(t)$ $j = 1, 2, \dots, k$ is exponential.

Proof

Observe that $\hat{F}(t) = e^{-\lambda t} = h(\hat{F}_1(t), \dots, \hat{F}_n(t))$ where h is the reliability function of a structure system \emptyset of order n and $\hat{F}_i(t)$ are NBAFR for $i=1, \dots, n$. Further, \emptyset has representation

$$\emptyset(x) = \min_{1 \leq j \leq p} \max_{i \in k_j} x_i$$

where p is the number of cut sets of the system, x_i is the state of the i th relevant component and k_j is the j th minimal cut set.

Therefore, if X_1, \dots, X_n are the lifetimes, then

$$e^{-\lambda x} = P\{\min_{1 \leq j \leq p} \max_{i \in k_j} x_i > x\}$$

or

$$(2) \quad e^{-\lambda x} = \sum_{j=1}^p P\{\max_{i \in k_j} X_i > x\}$$

Taking the derivative we get

$$\lambda e^{-\lambda x} = e^{-\lambda x} \sum_{j=1}^p \left\{ \frac{d}{dx} P[\max_{i \in k_j} X_i > x] / P\{\max X_i > x\} \right\}$$

Now by setting $x = 0$, we obtain

$$(3) \quad \lambda = \sum_{j=1}^p r_{k_j}(0)$$

where γ_{k_i} is the failure rate function of $\max_{i \in k_j} X_i$. On the other hand if the X_i 's are independent NBAFR then $\max_{i \in k_j} X_i$ is also NBAFR [2, p. 42]. Therefore (2) implies

$$e^{-\lambda x} \leq e^{-\left(\sum_{j=1}^p \gamma_{k_j}(0)\right) x}$$

or equivalently $\lambda \leq \sum_{j=1}^p \gamma_{k_j}(0)$. But this contradicts (3) unless

$$P \left\{ \max_{i \in k_j} X_i > x \right\} = e^{-\tau_{k_j}(0)x} \text{ for all } j = 1, 2, \dots, p$$

Now $\tau_{k_j}(0) = 0$ for k_j with more than one element. Also since

$\lambda = \sum_{j=1}^p \gamma_{k_j}(0)$, we should have at least one k_j with only one element. Therefore, we

should have (1) to be valid.

The above theorem improves Theorem 1 of Block and Savits in the sense that the assumption that the components are IFRA is replaced by the weaker assumption that the components are NBAFR. Also, as the NBU class of life distributions is contained in the class of NBAFR life distributions, it follows that if one replaces the assumption that the components are NBAFR with that they are NBU then the result still holds. This gives the observation of Block and Savits [4].

Theorem 2

If the convolution of n NBAFR distributions is exponential then $(n-1)$ of the distributions are degenerate at zero and the other distribution is exponential.

Proof

It is obvious that if the result is true for $n=2$ then it is true for any positive integer n . Now suppose that F and G are two NBAFR distributions and their convolution is exponential. Then at least one of them is absolutely continuous. Let F be absolutely continuous. Suppose G has mass $p > 0$ at zero then this implies G is degenerate at zero. This is so since G is NBAFR implies $-\log \hat{G}(x) \geq Lx$ where $L = -\lim_{x \rightarrow 0} [\log \hat{G}(x)/$

$x] = \infty$. Thus $-\log \hat{G}(x) = \infty$ for all $x > 0$ which is true only when $G(x) = 0$ for all x . Thus G is degenerate at zero. Now if G has a mass at zero then the theorem is obvious. Suppose that $G(0) = 0$. The fact that F is absolutely continuous and it is NBAFR imply that density f of F exists and is bounded near zero. Therefore if $F * G(x) = 1 - e^{-\lambda x}$ then we have

$$\lambda e^{-\lambda x} = \int_0^x f(x-y) dG(y) \leq bG(x)$$

for all x sufficiently small. Taking the limit as x tends to zero, we get $\lambda = 0$ which is impossible unless the exponential is degenerate. This implies that G should have a mass at zero which in turn yields G is degenerate at zero.

Remark 2

Theorem 2 is an improved version of Theorem 2.8 of Block and Savits [4]. A generalization of their result can also be found in Kitchen and Proschan [6]. We may mention here that Amed and Alzaid [7] proved similar result for the class of harmonic new better than used in expectation (HNBUE). However their result does not cover the NBAFR case since the NBAFR (HNBUE) class of distributions is not contained in the class of HNBUE (NBAFR) distribution.

Remark 3

It is obvious from the proof of the Theorem 2 that the following two statements are true: If the convolution of n NBAFR distributions is absolutely continuous with positive density at zero, then $(n-1)$ of the distributions are degenerate at zero and the remaining distribution is the same as the convolution. Also the assumption that F is NBAFR can be replaced with F has a bounded density at zero.

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أنظمة بحياة أسية وحياة مركباتها ذات ج ف ق م ف

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ملخص البحث. لقد استخدمنا فرض التوزيعات الأسية للحياة المكونة من وحدات ذات ج ف ق م ف لاثبات (أ) إذا كان النظام مطرد عندئذ يجب أن يكون النظام المتوال بوحدات أسية الحياة، (ب) إذا كانت حياة النظام عبارة عن مجموع حياة وحداته عندئذ فإن $(n-1)$ من الوحدات متكررة (منحلة) في نقطة الصفر والوحدة الباقية أسية.