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Civil Engineering

Versatile Finite Strip Method for Plate Bending

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Abstract. A versatile finite strip model for plate bending capable of dealing with plates continuous over rigid supports, in both directions, is developed. The model considers a rectangular finite strip with two degrees of freedom at each of the four corner nodes. The displacements in x-direction are represented by polynomials while in y-direction by specially developed series functions which afford the desired continuity of the plate over rigid supports. The interpolation surfaces for each degree of freedom are defined as products of appropriate functions in each direction. Beyond this, virtual work formulation yields the stiffness matrix equation of the structure. The results of analysis from the model are compared with those available in literature and ones obtained from finite element models. The method affords accurate results with a coarse mesh size.

Nomenclature

a	Span of a plate over a bay and finite strip span
b	Width of a finite strip
E	Modulus of elasticity
[f]	Interpolation functions matrix of the i^{th} series term
k	Constant of weighting function
M_x, M_y, M_{xy}	Internal moments of a plate
p	Loading intensity
$\{q\}_i$	Nodal displacement vector of the i^{th} sereis term
t	Plate thickness

$w(x,y)$	Deflection at a generic point
X_i	Beam shape functions in x-direction, $i=1$ to 4.
x, y, z	Cartesian coordinates of frame of reference
Y_{1i}, Y_{2i}	Series functions in y-direction of the i^{th} series term
α_i, μ_i	Mode constants
θ_x	Rotations about x-axis
θ_{xy}	Warping rotation in xy-plane
ν	Poisson's ratio
EFSM	Extended finite strip method
FEM	Finite element method
FSM	Finite strip method
VFSM	Versatile finite strip method

Introduction

As originally conceived [1], the finite strip method (FSM) only deals with a rectangular plate structure which has predefined boundaries along two parallel edges (boundary edges) and no continuity over them. The method subdivides the plate into strips by 'nodal-lines' normal to the boundary edges and considers amplitudes of displacement along and rotation about the nodal line as the degrees of freedom. The displacement of a generic point is expressed as a product of a series function, in the direction of a nodal line, which a priori satisfies the boundary conditions and a polynomial in the other direction.

A modification to the method was presented by Siddiqi and Vallabhan [2,3,4] as the extended finite strip method (EFSM) which considers rigid body displacements of the boundary edges and, therefore, allows inclusion of stiffeners along these edges. EFSM affords analysis of cantilever, corner supported plates, and plates carried on elastic supports along the edges or at corners.

Ref. [5] employes FSM to indirectly analyze a plate structure, continuous over the boundary edges, through flexibility approach. Ref. [6] covers a wide range of application of FSM to various types of structures including curved plates. Ref. [7] introduces a compound FSM to analyse curved plates over non-rigid supports.

As it is, FSM has inherent limitation built into it by the series functions used, which do not allow it to directly analyze the plates which are continuous over the boundary edges. FSM defines a strip element by nodal lines and uses displacement and nodal amplitudes as degrees of freedom.

Versatile Finite Strip Method (VFSM)

General

The proposed VSFM defines a finite strip by four nodes instead of two nodal lines of FSM, uses nodal degrees of freedom instead of amplitudes along a nodal line and shape- or interpolation-surfaces instead of series functions. In other words, this approach is exactly the same as that of finite element method (FEM) but with a major difference in the shape function in the long direction of strip.

Geometry

Fig. 1 shows a plate structure which may be continuous on either or both sides of the rigid supports parallel to x-axis. The geometry of strip is defined by length a in y-direction, width b in x-direction and thickness t in z-direction which is constant over the strip and can vary from strip to strip.

Degrees of Freedom

At each of the nodes there are two degrees of freedom, the rotation about x-axis (θ_x) and warping rotation [2] about z-axis (θ_{xy}). The displacement in the x, y, and z directions and rotation about y-axis at each node are absent and, therefore, the corresponding degrees of freedom are not defined.

Interpolation Surfaces

An interpolation surface (or function), for a degree of freedom, is expressed as a product of two independent functions, one in x-direction and the other in y-direction (variable separables).

The y-direction part of the interpolation function is a series of continuously differentiable functions of trignometric and hyperbolic nature while the x-direction part are the so called 'beam shape functions' which are differentiable and have linearly varying second order derivative. The interpolation functions, for a prismatic member, are evolved so that when i^{th} generalized displacement, q_i, is unity the rest of these displacements vanish and that second derivative (curvature) of the function, at the (near) node under consideration, is opposite in sign and twice as much as at the opposite (far) node.

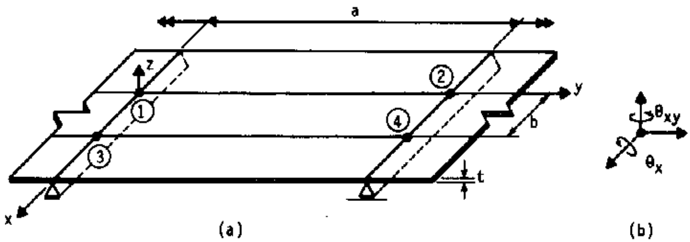

Fig. 1. Geometry of finite strip and the nodal degrees of freedom.

X-Functions

X-part of the shape functions used are same as the beam functions,

$$
\begin{aligned}
X_1 &= (2x^3 - 3x^2b + b^3) / b^3 \\
X_2 &= (x^3b - 2x^2b^2 + xb^3) / b^3 \\
X_3 &= (-2x^3 + 3x^2b) / b^3 \\
X_4 &= (x^3b - x^2b^2) / b^3
\end{aligned}
\tag{1}
$$

Y-Functions

Y-part of the shape functions are specially developed for the proposed model. Considering the function [4]

$$Y_{li} = \sin(\mu_i\, y/a) - \alpha_i \sinh(\mu_i\, y/a) \tag{2}$$

where

$$
\begin{aligned}
i &= 1, 2, 3\ldots, m \text{ (the series terms)}, \\
\alpha_i &= \sin\mu_i/\sinh\mu_i, \text{ and} \\
\mu_i &= 3.9266, 7.0685, 10.2102, \ldots, [(4m+1)/4]\pi
\end{aligned}
$$

which has boundary conditions, $Y(0) = Y''(0) = Y(a) = Y'(a) = 0$ and $Y'(0) \neq 1$, can be modified by a weighting function to create in it attributes required of an interpolation function mentioned earlier. Employing a weighting function [8] of $(a/\alpha_i\, u_i)$ $[1-(ky/a)]$ where value of k is determined through trials to be 0.7935 which yields the desired end conditions of $Y'(0) = 1$, $Y(0) = Y(a) = Y'(a) = 0$, $Y''(0) = -2Y''(a)$. The first term of the weighting function modifies the slope at $y = 0$, while the second modifies the curvatures. The i^{th} mode of function Y_1, therefore is,

$$Y_{1i} = \frac{a}{\alpha_i \ \mu_i} [\sin (\frac{\mu_i \ y}{a}) - \alpha_i \sinh (\frac{\mu_i \ y}{a})] \ (1 - \frac{ky}{a}) \qquad (3)$$

On the same lines, the i^{th} mode of the second function Y_2 which has end conditions of $Y'(a) = 1$, $Y(0) = Y'(0) = Y(a) = 0$, and $Y''(a) = -2Y''(0)$ is,

$$Y_{2i} = \frac{a}{(1 - \alpha_i) \ \mu_i} [- \sin (\mu_i - \frac{\mu_i \ y}{a}) + \alpha_i \sinh (\mu_i - \frac{\mu_i \ y}{a})] \ (\frac{ky}{a} - k + 1) \qquad (4)$$

The above functions are orthogonal and as such afford independent evaluation of contribution from an i^{th} term of the series.

XY-Function

Fig. 2 shows the interpolation surfaces for the two degrees of freedom at node-1 as examples.

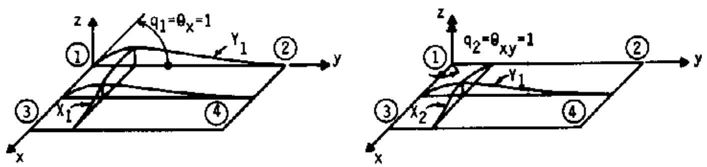

Fig. 2. Interpolation surface of generalized displacement of node-1

The interpolation surfaces f_{ni} for the eight degrees of freedom, in a row vector, corresponding to i^{th} series term are,

$$[f]_i = [X_1Y_{1i} \ \ X_2Y_{1i} \ \ X_1Y_{2i} \ \ X_2Y_{2i} \ \ X_3Y_{1i} \ \ X_4Y_{1i} \ \ X_3Y_{2i} \ \ X_4Y_{2i}] \qquad (5)$$

Therefore, displacement w(x,y) of a generic point on the strip may be obtained in terms of the generalized nodal displacements by superposition as,

$$w(x,y) = \sum_{i=1}^{m} [f]_i \ \{q\}_i \qquad (6)$$

where m is the number of series terms or modes considered.

Stiffness Matrix Equation

Using the plate bending equations and virtual work formulation, the stiffness matrix equation of a plate structure is obtained for the assumed displaced shape of Eq. (5). Computer implementation of the model employs the standard procedure of numerical intergration, half band assembly and Gaussian solution. Results of analysis of two examples are presented and compared with Timoshenko [9] value and solutions obtained from MZC [10,11] and BFS [12] finite element algorithms.

Example (1)

A single panel square plate, with all edges simply supported, carrying uniformly distributed load is analyzed. Results of deflections and moments are presented in Tables 1 to 3.

Table 1. Mid-point deflection of example (1) for different meshes (m)

Series terms	VFSM			MZC [10,11]			BFS [12]		
	Mesh	Result	% Error	Mesh	Result	% Error	Mesh	Result	% Error
1	4×1	-0.9539×10^{-3}	+0.86	4×4	-1.0090×10^{-3}	+6.68	4×4	-0.9471×10^{-3}	+0.14
	6×1	-0.9537×10^{-3}	+0.83	6×6	-0.9742×10^{-3}	+3.00	6×6	-0.9467×10^{-3}	+0.10
	8×1	-0.9537×10^{-3}	+0.83	8×8	–	–	8×8	–	–
2				Timoshenko [9] Value : Deflection = -0.9458×10^{-3}					
	4×1	-0.9477×10^{-3}	+0.20						
	6×1	-0.9475×10^{-3}	+0.18	E = 3×10 kN/m^2					
	8×1	-0.9475×10^{-3}	+0.18	ν = 0.3					
3	4×1	-0.9450×10^{-3}	+0.09	t = 0.25 m					
	6×1	-0.9448×10^{-3}	−0.11	a = b = 10 m					
	8×1	-0.9448×10^{-3}	−0.11	p = −1 kN/m^2					

Note: Percent error compared with Timoshenko value.

Table 2. Mid-point M_x in plate of example (1) for different meshes (kN–m)/m

Series terms	VFSM			MZC [10,11]			BFS [12]		
	Mesh	Result	% Error	Mesh	Result	% Error	Mesh	Result	% Error
1	3×1	5.029	+4.99	3×3	4.582	−4.34	3×3	4.691	−2.07
	5×1	5.043	+5.28	5×5	4.722	−1.42	5×5	4.755	−0.73
	7×1	5.047	+5.37	7×7	4.757	−0.69	7×7	4.773	−0.35
2				Timoshenko [9] Value : 4.790					
	3×1	4.843	+1.11						
	5×1	4.858	+1.42						
	7×1	4.862	+1.50						
3	3×1	4.647	−2.99						
	5×1	4.663	−2.65						
	7×1	4.667	−2.57						

Note: Percent error compared with Timoshenko value.

Table 3. Mid-point M_y in plate of example (1) for different meshes (kN–m)/m

Series terms	VFSM			MZC [10,11]			BFS [12]		
	Mesh	Result	% Error	Mesh	Result	% Error	Mesh	Result	% Error
1	3×1	5.029	+4.99	3×3	4.582	−4.34	3×3	4.691	−2.07
	5×1	5.043	+5.28	5×5	4.722	−1.42	5×5	4.755	−0.73
	7×1	5.047	+5.37	7×7	4.757	−0.69	7×7	4.773	−0.35
2	3×1	4.843	+1.11	Timoshenko [9] Value : 4.790					
	5×1	4.858	+1.42						
	7×1	4.862	+1.50						
3	3×1	4.647	−2.99						
	5×1	4.663	−2.65						
	7×1	4.667	−2.57						

Note: Percent error compared with Timoshenko value.

Example (2)

A two-bay, two-span plate with different span lengths and thicknesses, simply supported along edge and intermediate supports shown in Fig. 3 is analysed by VSFM and BFS – algorithm using 8 × 2 and 8 × 7 mesh size respectively. The results of analyses along lines AA and BB are plotted in Figs. 4 to 7. The resultant stresses are evaluated along mid-line (line BB) of an element. $E = 3 \times 10^5$ k/sft, $\nu = 0.3$ and $p = -0.6$ k/sft.

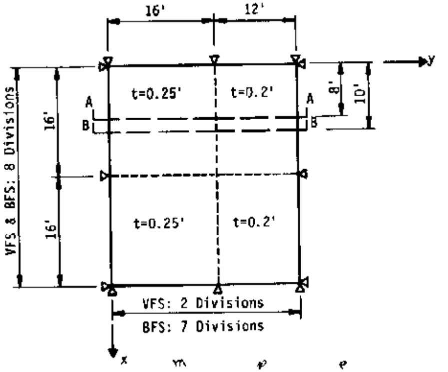

Fig. 3. Multispan, multibay plate of example (2)
(Legend: ▷——◁ hinged edge).

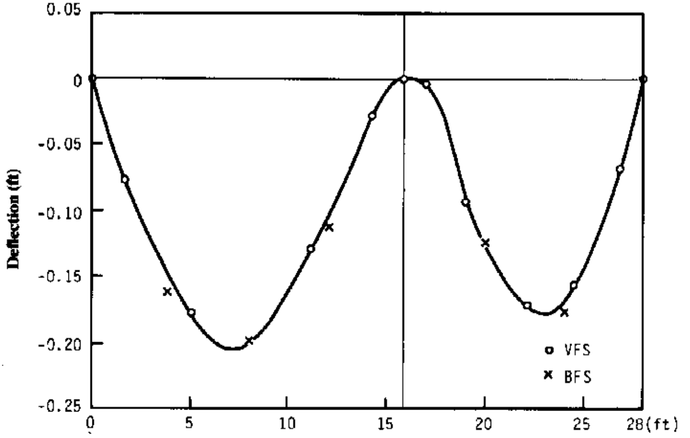

Fig. 4. Plate of example (2), deflection along section A-A

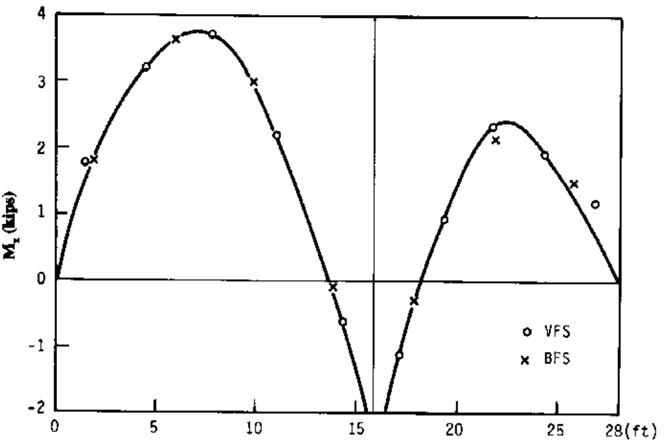

Fig. 5. Plate of example (2), M_x along section B-B

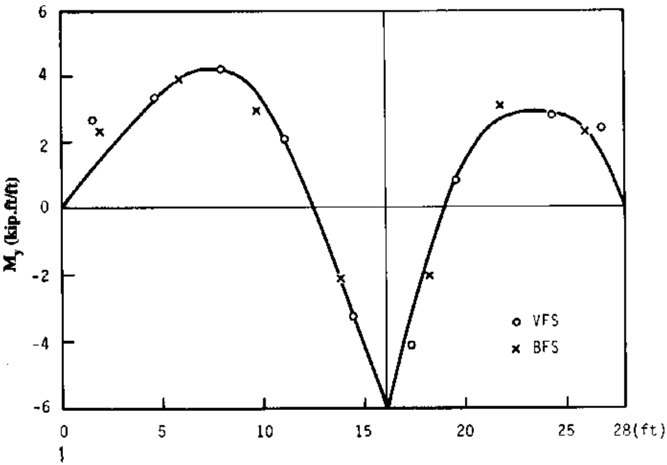

Fig. 6. Plate of example (2), M_y along section B-B

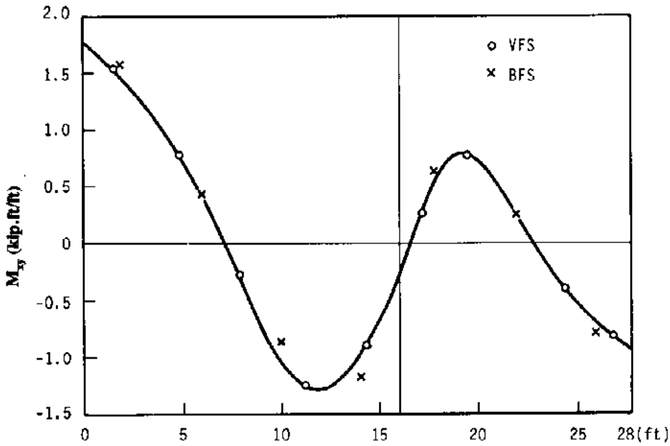

Fig. 7. Plate of example (2), M_{xy} along section B-B

Conclusions

The versatile finite strip method is developed for analysis of strip-wise prismatic structures which are simply or continuously supported over rigid supports. The model employs finite strip elements, which span over the full length (in y-direction) of a bay, have four nodes and two degrees of freedom per node viz. rotation about x-axis and warping rotation in xy-plan. The interpolation functions for a degree of freedom are expressed as product of a polynominal in the x-direction and a specially formulated series function in the y-direction. The number of unknowns involved in the method is small for equally accurate results obtained from finite element models. The contributions made by the second and the third series terms of VSFM are small and do not necessarily enhance the accuracy of end results. The first series term, for all practical purposes, affords adequately accurate analysis.

In VSFM, as in most finite element models, the values of stress resultants, evaluated near boundaries of an element, are less accurate.

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طريقة الشريحة المحدودة للبلاطات تحت تأثير الانحناء

غلام حسين صدقي وفيصل المشاري

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ملخص البحث. لقد تم تطوير نموذج عام للشريحة المحدودة لتحليل البلاطات المستمرة في اتجاهين على مساند ثابتة. ويتضمن النموذج شريحة مستطيلة محدودة تتحرك أركانها الأربعة في اتجاهين. وتحدد الإزاحات في الاتجاه السيني بدوال متعددة الحدود وفي الاتجاه الصادي بدوال متسلسلة. وباستخدام طريقة الشغل الافتراضي تم تكوين معادلة مصفوفة الصلابة الإنشائية. ولقد تمت مقارنة التحليل باستخدام النموذج مع نتائج نماذج طريقة العنصر المحدود. ولقد أعطت طريقة النموذج المقترح نتائج أكثر دقة عند استخدام شرائح صغيرة الحجم.