

MECHANICAL ENGINEERING

Numerical Solutions for Unsteady One-Dimensional Flow Problems

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(Received 26/9/1989; Accepted for publication 10/6/1990)

Abstract. The effectiveness of various numerical schemes in solving the one-dimensional wave equation, one-dimensional gas dynamics equation and Burgers model of turbulence is studied. The accuracy of the solutions is gauged by comparing them with the exact analytical solutions. The effects induced by changes in step size are discussed.

Introduction

Numerical solutions of the wave equation in one Cartesian space dimension are important in the engineering analysis of pressure transients in piping systems carrying liquids. Since the equation is hyperbolic, the customary method is an explicit marching procedure in time and space, based on the method of characteristics, as discussed by Wylie and Streeter [1]. Difference approximations for the hyperbolic equations work well when the solution is smooth. However, the solution of hyperbolic Partial Differential Equations (PDE) can include discrete jumps. (This is contrary to the solution of elliptic or parabolic Partial Differential Equations, which are always continuous in space and time). For example, in a pipe flow of some chemicals, the concentration of chemical species can have a sudden jump. The wave equation is an adequate mathematical model of the physical problem for the limiting case of frictionless flow; when viscous effects are important, the one dimensional wave equation no longer describes the problem. Additional viscous terms and an additional transverse momentum equation may become necessary, with the degree of complexity depending on the exactness with which the viscous effects are described.

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Burgers equation which is a One-dimensional analog to the unsteady Navier-Stokes equation and which is often used to test numerical methods and theoretical closures, can be solved exactly in terms of standard functions of mathematical physics by means of the Cole-Hopf [2,3] transformation, provided:

- i) The initial conditions for Burgers equation maps under the inverse Cole-Hopf transformation into mathematically tractable initial conditions for the heat conduction equation.
- ii) The boundary conditions for Burgers equation are either to be imposed as, say, asymptotically vanishing or constant on infinite domains or have counter parts which make for tractable boundary value problems for the heat conduction equation [4].

Burgers devoted most of his life to the analysis of the equation, $u_t + u u_x = \nu u_{xx}$, which bears his name [5; pp. 171-199], including its statistical properties.

In this paper, we evaluate the effectiveness of the various numerical schemes in solving One-dimensional wave equation, gas dynamics equation and Burgers model of turbulence. The effects induced by changes in step size are discussed.

Wave Propagation in One Space Dimension

There are several difficulties involved in the solution of the wave equation. In the finite difference method, the major difficulties encountered are: i) the phase speed of a wave changes especially when the mesh is non-uniform. The wave may not propagate at all if the mesh size changes too abruptly. ii) The amplitude may not be maintained as the wave propagates. iii) the wave may travel in the wrong direction and iv) sharp signals may not be retained.

To illustrate the behaviour of the discontinuous solution of a hyperbolic PDE, we consider

$$u_t + a u_x = 0 ; x \geq 0, t \geq 0 \quad (1)$$

The initial and boundary conditions are

$$u(x,0) = \begin{cases} 1 & x \leq 1 \\ 0 & x > 1 \end{cases} \quad (2a)$$

$$\text{and } u(0,t) = 0 \quad t > 0 \quad (2b)$$

The analytical solution for this problem is

$$u(x,t) = u(x-at, 0) \quad (3)$$

This solution represents the wave of a rectangular shape travelling at the speed a .

Time and Space Differencing

Number of different numerical schemes can be derived depending on the difference approximation selected for each of u_t and u_x , Roache [6]. When $a > 0$ (flow is in the positive direction) the back ward difference approximation

$$u_x \approx (u_i - u_{i-1})/\Delta x \quad (4a)$$

if, $a < 0$, the forward difference approximation

$$u_x \approx (u_{i+1} - u_i)/\Delta x \quad (4b)$$

where Δx is the grid interval in space.

The time difference approximation is

$$u^{(n+1)} \approx u^{(n)} + (\Delta t) (u_t) + O(\Delta t^2) \quad (5)$$

Here Δt is the discrete time increment.

I) Forward time and backward space (FTBS) difference scheme

FTBS approximation for the equation (1) is represented as

$$\frac{u_i^{(n+1)} - u_i^{(n)}}{\Delta t} + a \frac{u_i^{(n)} - u_{i-1}^{(n)}}{\Delta x} = 0 \quad (6a)$$

Equation (6a) can be written as

$$\frac{u_i^{(n+1)} - u_i^{(n)}}{\Delta t} + a \frac{u_{i+1}^{(n)} - u_{i-1}^{(n)}}{2 \Delta x} - |a| \Delta x \frac{u_{i+1}^{(n)} - 2 u_i^{(n)} + u_{i-1}^{(n)}}{2(\Delta x)^2} = 0 \quad (6b)$$

In the equation (6b), the second term is the central difference approximation for u_x and the third term is a central difference approximation for $-|a| \Delta x u_{xx}/2$. The FTBS scheme as using the central difference for u_x and artificially adding the central difference approximation for $-|a| \Delta x u_{xx}/2$, which is named "numerical viscosity term".

When the initial and boundary conditions are all non-negative, the solution of the present scheme never becomes negative.

II) Lax-wendroff Scheme

Equation (1) is written in this scheme as

$$u_i^{(n+1)} = u_i^{(n)} - (Y/2) [u_{i+1}^{(n)} - u_{i-1}^{(n)}] + (Y^2/2) [u_{i-1}^{(n)} - 2u_i^{(n)} + u_{i+1}^{(n)}] \quad (7)$$

where $Y = a \Delta t / \Delta x$.

III) Third-order upwind scheme

The spatial derivative can be approximated by the Third-order accurate difference approximation given by [7]

$$a(u_x)_i \approx \begin{cases} a(u_{i+2} - 2u_{i+1} + 9u_i - 10u_{i-1} + 2u_{i-2})/6\Delta x & \text{for } a > 0 \\ a(-2u_{i+2} + 10u_{i+1} - 9u_i + 2u_{i-1} - u_{i-2})/6\Delta x & \text{for } a < 0 \end{cases} \quad (8)$$

which can be written in a single form as

$$a(u_x)_i \approx a(-u_{i+2} + 8u_{i+1} - 8u_{i-1} + u_{i-2})/12\Delta x + |a| (u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i-2})/4\Delta x \quad (9)$$

The difference approximation for Equation (1) is written as

$$u_i^{(n+1)} = u_i^{(n)} + \Delta t [-a(-u_{i+2}^{(n)} + 8u_{i+1}^{(n)} - 8u_{i-1}^{(n)} + u_{i-2}^{(n)})/12\Delta x - |a| (u_{i+2}^{(n)} - 4u_{i+1}^{(n)} + 6u_i^{(n)} - 4u_{i-1}^{(n)} + u_{i-2}^{(n)})/4\Delta x] \quad (10)$$

Equation (10) has no aliasing error due to a third derivative term.

One-Dimensional Gasdynamics Equation

The unsteady, inviscid equation of motion is

$$u_t + u u_x = 0 \quad (11)$$

subject to the following conditions

$$u(x,0) = x \quad ; 0 \leq x \leq 1 \quad (12a)$$

$$u(0,t) = 0 \quad ; \quad t \geq 0 \quad (12b)$$

An analytical solution of equation (11) is

$$u(x,t) = x/(1+t) \quad (13)$$

One can also present the finite difference approximations for equation (11) for the schemes (I-III) mentioned in the previous section.

The Burgers Turbulence Model

The equation of motion is

$$u_t + u u_x = \nu u_{xx} \quad (14)$$

where u is the net velocity, ν is the kinematic viscosity. Equation (14) is subject to the following initial and boundary conditions

$$u(x,0) = \sin \pi x \quad ; 0 \leq x \leq 1 \quad (15a)$$

$$u(0,t) = u(1,t) = 0 \quad ; t \geq 0 \quad (15b)$$

The theoretical solution of equation (14) subject to boundary conditions (15a,b) was obtained by Cole[2] and compiled by Benton and Platzmann [8]

$$u(x,t) = 4\pi\nu \left[\sum_{n=1}^{\infty} n a_n \exp(-n^2 \pi^2 \nu t) \sin n\pi x / (a_0 + 2 \sum_{n=1}^{\infty} a_n \exp(-n^2 \pi^2 \nu t) \cos n\pi x) \right] \quad (16)$$

where $a_n = (-1)^n I_n(\frac{1}{2}\pi\nu)$ and $I_n(z)$ denotes the modified Bessel functions of first kind. This analytical solution is numerically untractable at small t and ν as $I_n(z)$ with z going to infinity behaves asymptotically as $e^z (2\pi z)^{-1/2}$; independent of n . For values of ν smaller than 0.01, the analytical formula (16) fails to produce any valid result because of the extremely slow rate of convergence of the infinite series contained in it. The values of these integrals have been computed by Ames [9]. A better form comes from the convolution product in Cole's transformation, *i.e.*

$$u(x,t) = \left[- \int_{-\infty}^{+\infty} \sin \pi (x-n) f(x-n) \exp(-n^2/4vt) dn / \int_{-\infty}^{+\infty} f(x-n) \exp(-n^2/4vt) dn \right] \quad (17)$$

with $f(y) = \exp(-\cos \pi y/2 \pi v)$

Using similar finite difference approximation as given in the previous sections, equation (14) was approximated by the following numerical schemes (for more details, Roache [6]):

a) Euler's explicit, b) Euler's implicit, c) Crank-Nicolson, d) MacCormack and e) Third order upwind scheme.

A comparison of the values of u obtained by the schemes (a-e) and those by other numerical methods [10] are listed in the Table 1.

Discussion

In order to evaluate the performance of various schemes we presented the variation of u for the wave equation using schemes (I-III) are shown in Figs. (1-3). The exact solution of equation (1) at any time has the same square shape as the initial distribution but the location of the wave is continuously advanced by the unit velocity. Throughout the tests, the grid spacing is $\Delta x=0.1$, $a=1.0$ but the time step is varied for each test.

The FTBS scheme never gives negative values of the numerical solution nor has any oscillatory behaviour. However, the waves tends to spread and be smeared as it travels. The height of the wave becomes lower and the width increases as shown in Fig. 1. The Lax-Wendroff scheme produces a much less smearing effects of waves than the FTBS scheme, but it has a significant oscillatory behaviour (Fig. 2), which is the aliasing error due to the third-order truncation error. This trend increases as $\gamma = a \Delta t / \Delta x$ approaches zero. with the third order upwind scheme, the wave is also distorted, but the height and width of the square wave are both maintained as shown in Fig. 3. One can note that smaller $\Delta t=0.0001$, scheme III (Fig. 3d), resembles the similar nature of Fig. 2 (a). However, for equal space and time intervals the scheme fails as shown in Fig. 3(a). Fig. 4 shows the results obtained for the gasdynamics equation(11) using the schemes (I-III) for various values of t with $\Delta x=\Delta t=0.05$.

Variations of u along the x -axis for the Burgers equation for $v = 1.0, 0.1, 0.01, 0.001$ and 0.0001 obtained by the MacCormack scheme is shown in Figs 5-8. Fig. 5 explains the variation of the velocity u for various values of time with equal space and time intervals. The details of the formation of a shock in the neighbourhood of $x=1.0$ are illustrated in Figs 6-8, for $v = 0.01, 0.001$ and 0.0001 respectively.

Table 1. Comparison of the values of u obtained by the schemes (d-h) and those obtained by (a-c) with $\Delta x = 0.01$, $\Delta t = 0.0025$ at (i) $x = 0.5$, $t = 0.05$; (ii) $x = 0.5$, $t = 0.25$.

γ	a	b	c	d	e	f	g	h
(i)	0.01	0.98337	0.98285	0.98286	0.97790	0.98146	0.98235	0.98299
	0.001	Failed	0.98701	0.98700	0.97643	0.97580	0.98232	0.98432
(ii)	0.01	0.79675	0.79706	0.79707	0.78441	0.78365	0.79215	0.79641
	0.001	Failed	0.80559	0.80550	0.78969	0.78887	0.79089	0.79316

a: Analytical, b: Matrix factorization (Ref. [10]), c: perturbed iterative scheme (Ref. [10]),
d: Euler's explicit, e: Euler's implicit, f: Crank-Nicolson, g: MacCormack, h: Third order upwind.

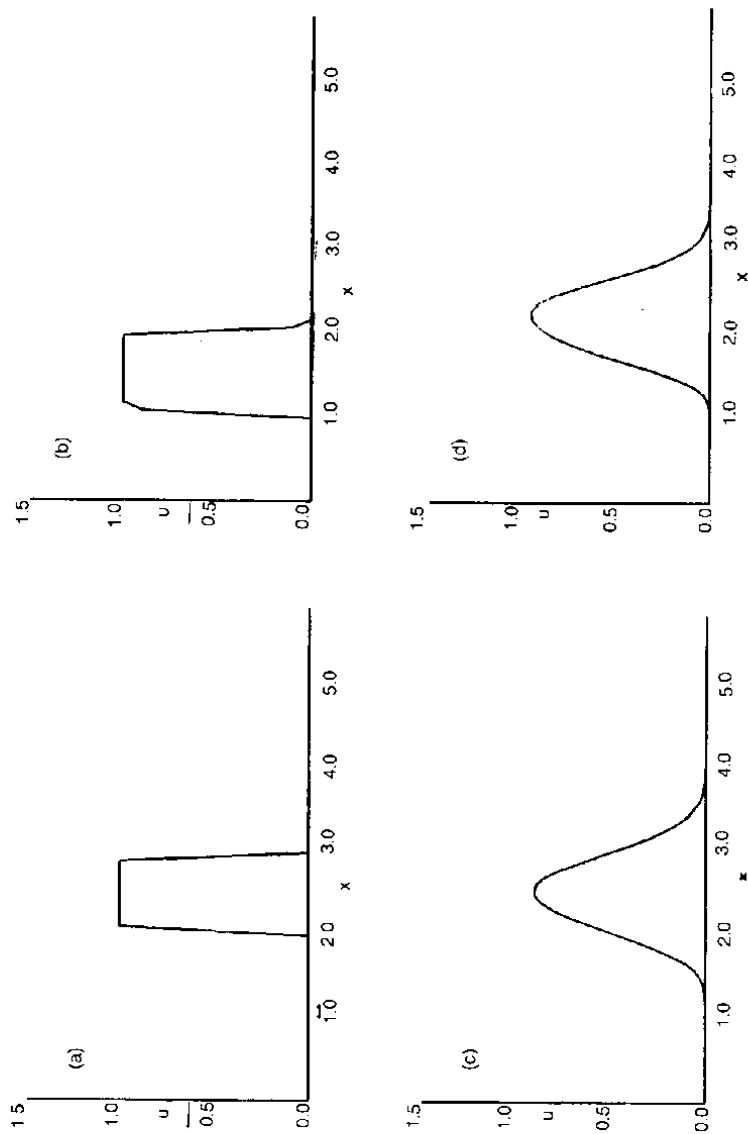


Fig. 1. Variation of u using scheme FTBS with: (a) $\Delta x = 0.1$ at $t = 1.0$; (b) $\Delta x = 0.1, \Delta t = 0.01$ at $t = 0.01$; (c) $\Delta x = 0.1, \Delta t = 1.0$ at $t = 1.0$; (d) $\Delta x = 0.1, \Delta t = 1.0$ at $t = 0.01$.

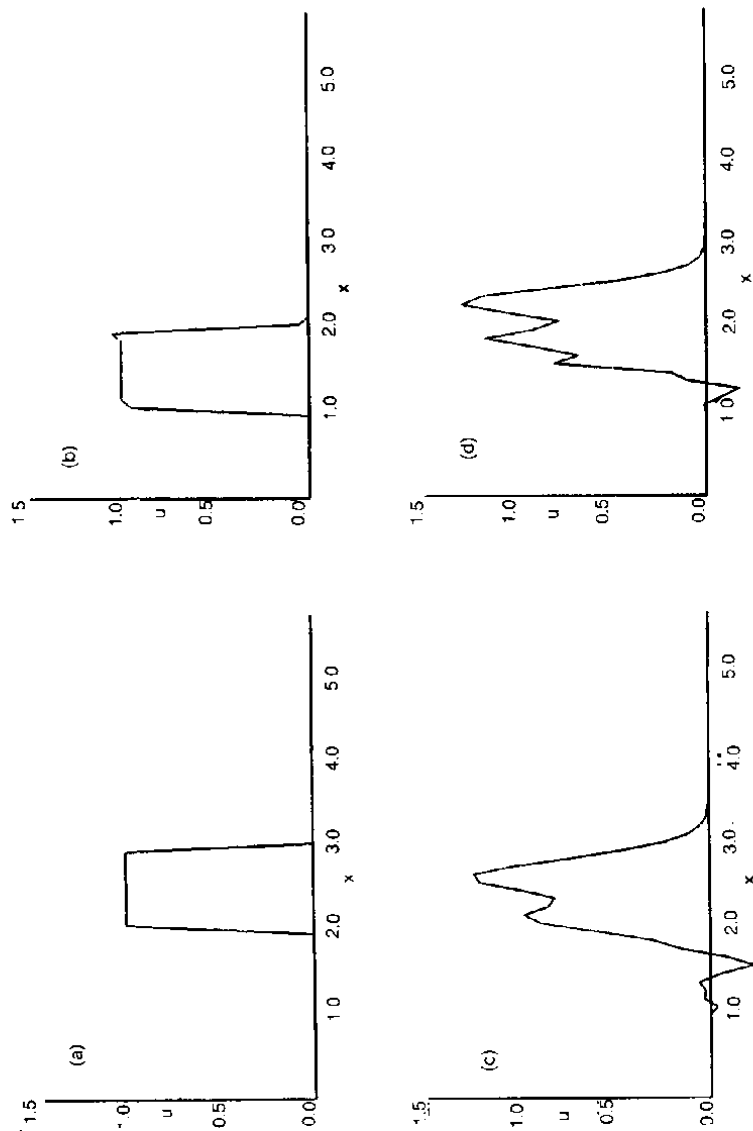


Fig. 2. Variation of u using scheme Lax-Wendroff with $\Delta x = 0.1$: (a) $\Delta t = 0.1$ at $t = 1.0$; (b) $\Delta t = 0.01$ at $t = 1.0$; (c) $\Delta t = 0.01$ at $t = 0.7$; (d) $\Delta t = 0.001$ at $t = 0.7$.

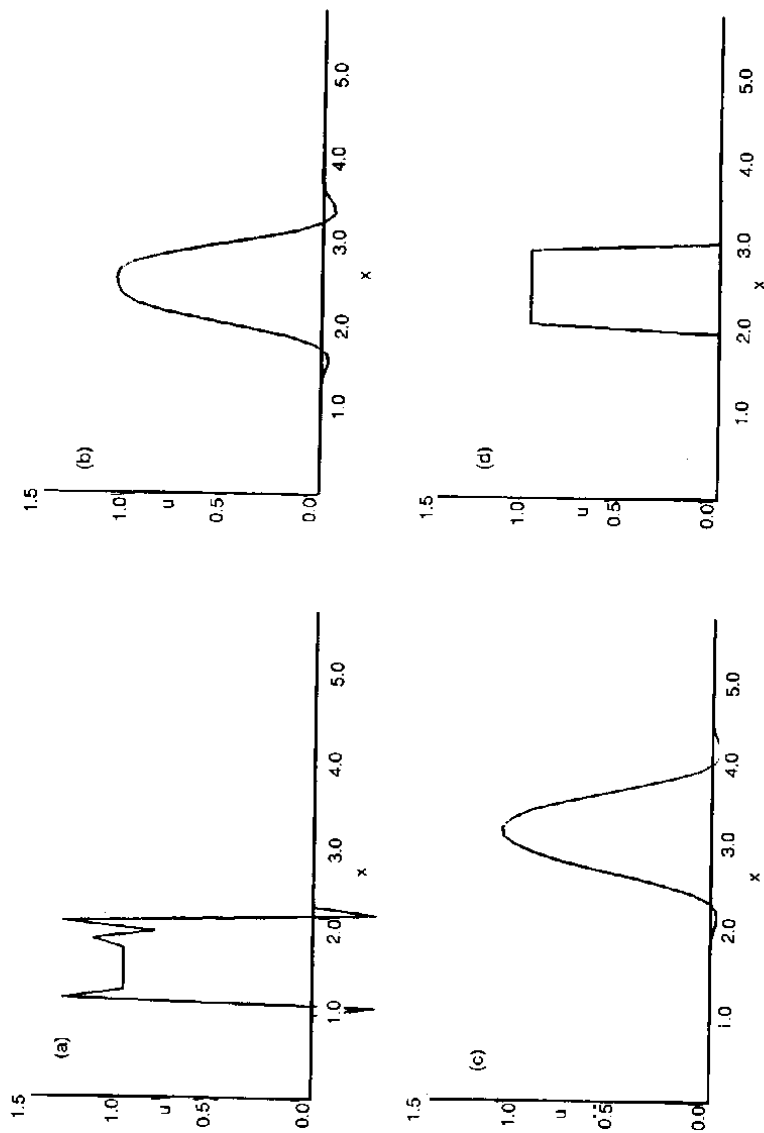


Fig. 3. Variation of u using third-order upwind explicit scheme with $\Delta x = 0.1$: (a) $\Delta t = 0.1$ at $t = 0.1$; (b) $\Delta t = 0.001$ at $t = 0.001$; (c) $\Delta t = 0.5$; (d) $\Delta t = 0.9$ at $t = 0.9$

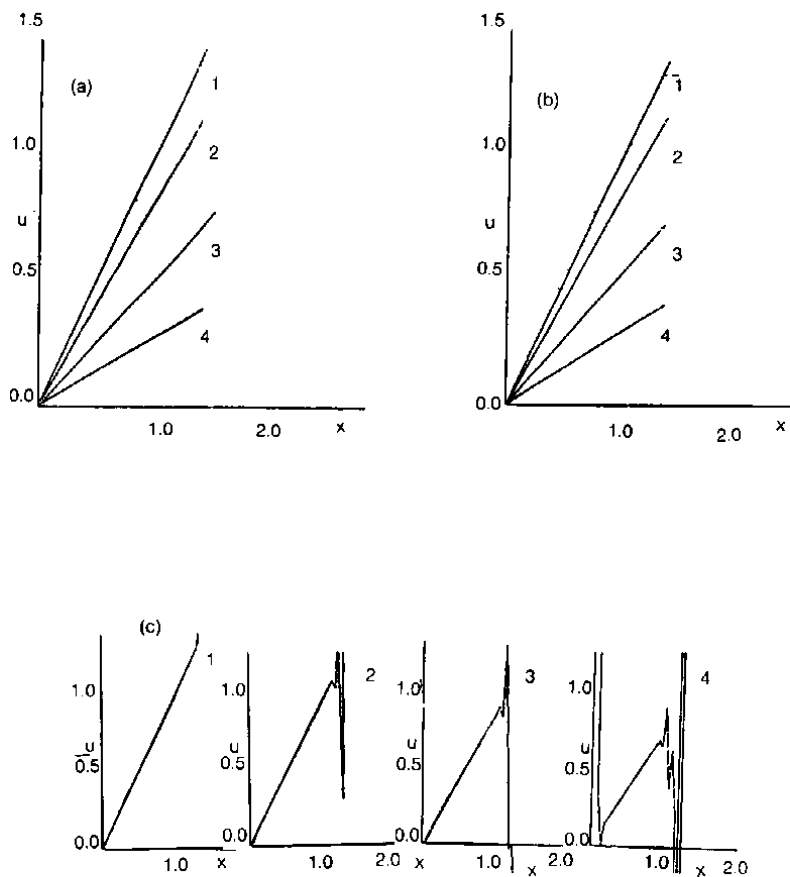


Fig. 4. Variation of u along x -axis for the gasdynamics equation: (a) FTBS; (b) Lax-wendroff; (c) Third-order-upwind explicit, at, (1) $t=0.05$, (2) $t=0.25$, (3) $t=1.0$, (4) $t=2.5$, with $\Delta x = \Delta t = 0.05$.

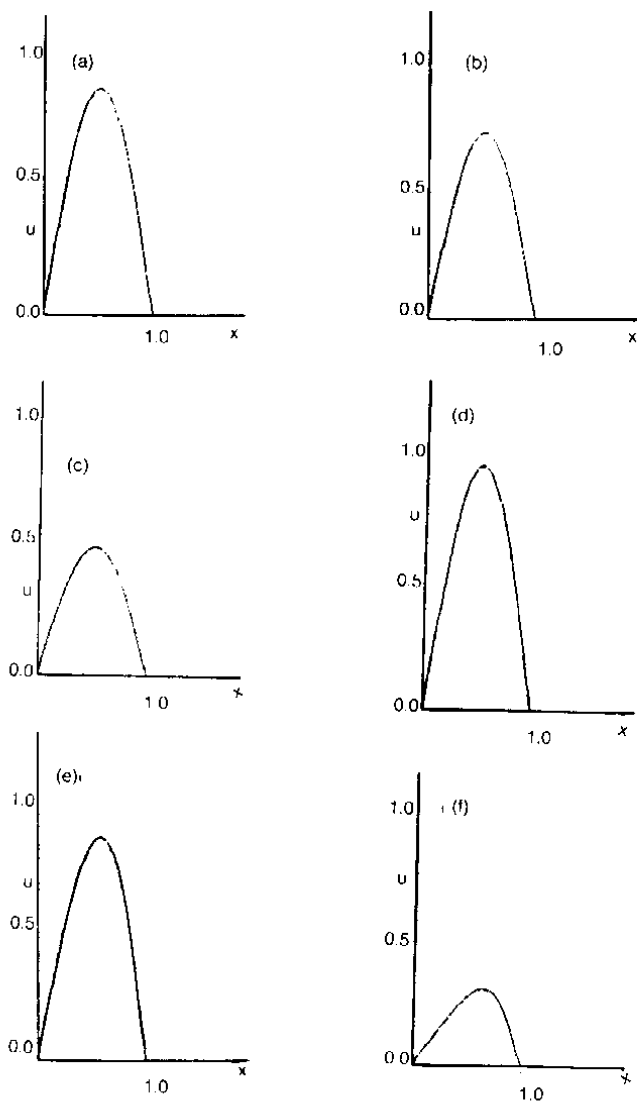


Fig. 5. Variation of u along the x -axis for the Burger's equation with $\gamma=1.0$, $\Delta x = \Delta t=0.01$: (a) $t=0.01$, (b) $t=0.03$, (c) $t=0.07$; and $\gamma=0.1$, $\Delta x = \Delta t=0.01$: (d) $t=0.01$, (e) $t=0.09$, (f) $t=1.0$.

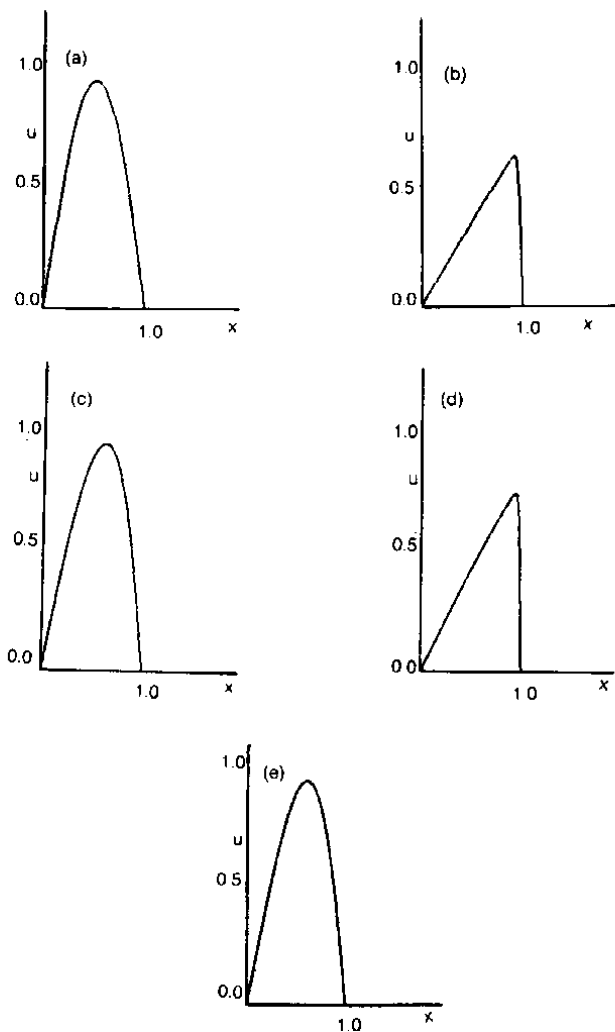


Fig. 6. Variation of u along the x -axis for the Burgers' equation with $\gamma = 0.01$, $\Delta x = \Delta t = 0.01$; (a) $t = 0.01$, (b) $t = 1.0$; $\Delta x = \Delta t = 0.001$ (c) $t = 0.1$, (d) $t = 0.8$; (e) $\Delta x = 0.0006$, $\Delta t = 0.001$ at $t = 0.1$.

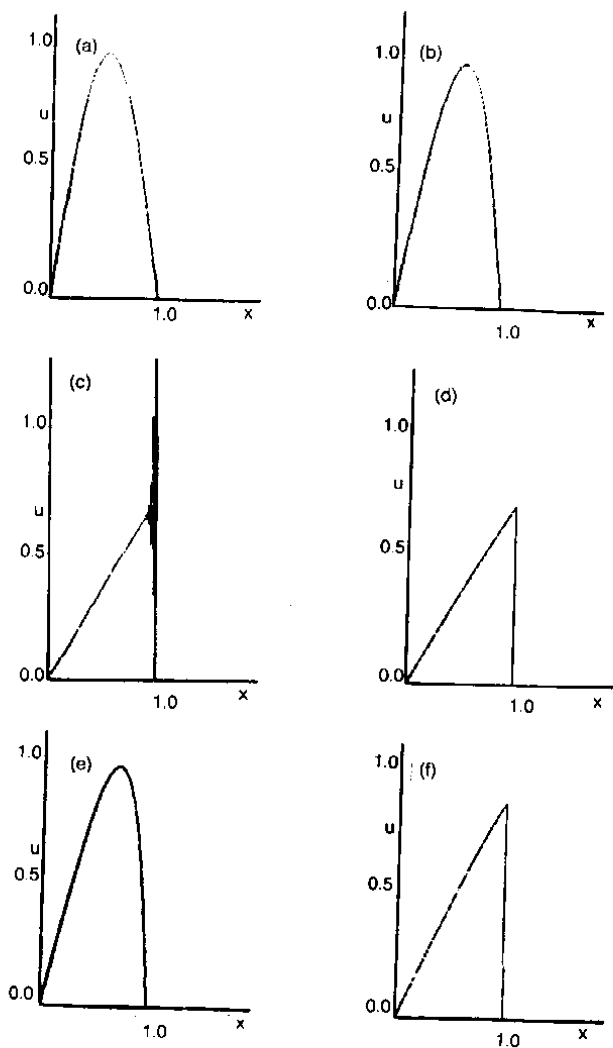


Fig. 7. Variation of u along the x -axis for the Burger's equation with $\gamma=0.001$, $\Delta x = \Delta t = 0.01$; (a) $t=0.01$, (b) $t=0.1$, (c) $t=1.0$, (d) $\Delta x=0.002$, $\Delta t=0.01$ at $t=1.0$; $\Delta x = \Delta t = 0.001$; (e) $t=0.1$ and (f) at $t=0.6$.

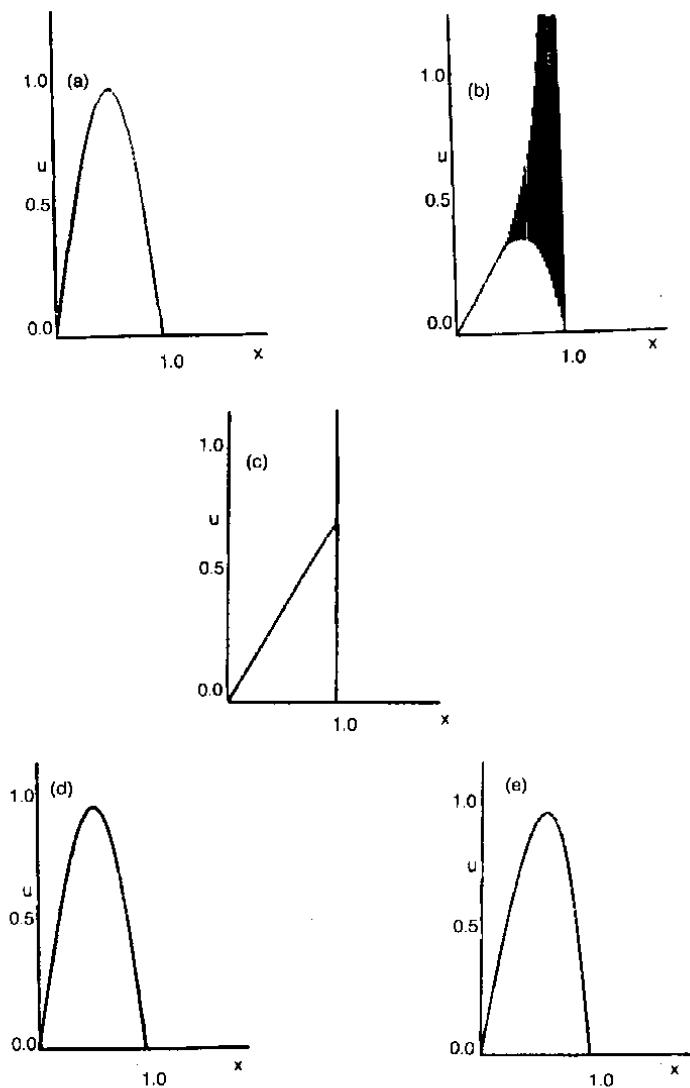


Fig. 8. Variation of u along the x -axis for the Burger's equation with $\gamma=0.0001$, $\Delta x = \Delta t=0.01$; (a) $t=0.01$, (b) $t=1.0$, (c) $\Delta t=0.01$, $\Delta x=0.001$ at $t=1.0$; $\Delta x=0.001$, $\Delta t=0.0001$; (d) at $t=0.01$ and (e) at $t=0.1$.

As the value of v is reduced, a definite and consistent pattern emerged on all the grids; the disturbances ($v=0.0001$) form in the vicinity of $x=0.5$ and propagate to the right with a steepened front as t increases.

The One-dimensional gas dynamics equation represents a pure inertial motion and the Burger equation represents an inertial motion with effects of viscosity, v . If $v=0$, the equation describes the inertial motion of a one-dimensional continuum until the time of formation of discontinuities or shocks. It is well known that the Burgers equation produces shocks quite well for simple flows. However, it is of interest (apparent paradox) that $v=0$ also allows shock formation even though $v \neq 0$ is physically required to generate discontinuities. This apparent paradox is intuitively similar to the success of potential flow predictions of lift and induced drag on airfoils and hydrofoils, where $v=0$. Potential flow predictions become better, correlated with experiments as the Reynolds number becomes very large.

Conclusion

Various numerical schemes in solving one-dimensional wave equation, gas dynamics equation and Burgers equation is studied. From this investigation, we conclude the following:

- 1) The FTBS scheme, is stable if γ (Courant number) ≤ 1 .
- 2) The Lax-Wendroff and MacCormack schemes are second-order accurate. The accuracy of the schemes are best when $\gamma = 1$. Therefore, accuracy decreases even when Δt is decreased unless Δx is also decreased.
- 3) The third order upwind scheme has an accuracy of third order in space but forward or backward Euler differencing in time introduces errors of second order. However, the magnitude of the second-order errors can be decreased by using a small Δt independently of Δx .

Acknowledgement. The authors wish to thank the referees for their valuable comments and suggestions which helped them to improve the presentation.

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الحلول العددية لمشاكل السريان غير المنتظم الأحادية البعد

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ملخص البحث. تم دراسة فاعلية الطرق العددية لحل معادلة الموجة ومعادلة ديناميكا الغازات الأحادية البعد ونموذج بيرجرز للاضطراب وقد قَدِّرت دقة الحلول بمقارنتها مع الحلول التحليلية الدقيقة ونوقشت التأثيرات التي تنتج عن التغيرات في طول الخطوة.