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On Non-linear Free Vibrations of a Beam with Pinned Ends

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Abstract. The free vibration of a pinned end beam undergoing large transverse deflection is examined. The equation of motion for this problem is reduced to a Duffing-type equation with a small perturbation parameter. The method of multiple scales is used to determine a uniformly valid higher third order perturbation solution. The predictions of the non-linear frequencies are in excellent agreement with the exact solution obtained from the direct integration of elliptic integral. Also presented are the harmonic contents of the non-linear transverse displacement.

Introduction

The analysis of non-linear vibrations of beams has received much attention in the literature. Various investigators have used approximate analytical methods as well as numerical methods such as finite difference and finite element methods. A comprehensive set of references of such work has been cited by Sathyamoorthy [1;2]. Ray and Bert [3] studied the stability of a pinned-end beam experimentally. Bennet and Eisely [4] investigated the stability of a simply supported beam excited at center and at quarter points. They used a multi-mode Galerkin approach to obtain a Duffing-type equation and then applied a perturbation technique to obtain a Mathieu-type equation. Tseng [5] used essentially the same procedure to investigate a beam with clamped ends. Bert and Fisher [6] used assumed mode analysis and Stoker's approximate equations [7, p. 189] for Mathieu stability regions to analyze the stability. Raju *et al.* [8] linearized the strain-displacement relations and applied Rayleigh-Ritz method to obtain a one term approximation for the amplitude-frequency relation. Sarma *et al.* [9] discussed various finite element formulations of large amplitude free vibrations of pinned end beam and presented an analytical formulation based on Rayleigh-Ritz method. Some controversial points were raised by many researchers on the assumptions introduced during the formulation using Rayleigh-Ritz method. Singh *et al.* [10] clarified some of these points. In another paper, Singh *et al.* [11] reduced the finite element matrix equations to a scalar second order differential equation of the Duffing-type which was solved by using direct integration. Essentially, they numerically integrated the elliptic function which resulted in exact values for the non-linear frequencies.

There is a wide discrepancy between the results of the approximate analytical solutions available in the literature and those of the finite element and finite difference solutions. Dumir and Bhaskar [12] discussed the source of errors in some finite element formulations which are traced back to the introduction of a linearizing constant function in the expression for the strain energy. Utilizing averaging techniques, Rao [13] obtained two comparable solutions each of first order. The first solution is obtained by assuming harmonic frequency (space model), while the second is obtained by assuming the linear fundamental mode shape (time model) and using the hybrid-Galerkin method. More recently, Pillai and Rao [14] examined the frequency - amplitude relation by using a time domain method, the Galerkin method and the method of harmonic balance. They stated incorrectly that "since the coefficients of non-linear terms in differential equation in general do not involve small parameter, the usual perturbation techniques are inappropriate".

The purpose of this paper is to clarify that the title problem can be formulated as a Duffing non-linear oscillator. A small perturbation prameter is introduced by using some transformations. Consequently, the problem can be analyzed by any of the available usual perturbation techniques discussed in any book on perturbation. One of these techniques, namely the method of multipe scales is used to determine a higher third order perturbation solution. It is shown that the method of multiple scales yields accurate predictions of the non-linear frequencies which are in excellent agreement with the exact solution obtained from the elliptic integral. Also presented are the harmonic contents of the non-linear transverse displacement.

Equations of Motion

Consider a slender beam of uniform cross section made of homogenous isotropic material without damping is pinned at its immovable ends. The equation of motion is [15]

$$EI\, w_{,xxxx} - N_x w_{,xx} + m\, w_{,tt} = 0, \tag{1}$$

where

$$N_x = N_0 + \frac{EA}{2L} \int_0^L w_{,x}^2\, dx. \tag{2}$$

A is the cross-sectional area, E the Young's modulus, I the moment of inertia, L the length of the beam, N_0 the initial axial tensile force, t the time, w the transverse deflection of the beam, x the axial coordinate, m the mass per unit length, and a comma denotes partial differentiation with respect to the varibles following the comma.

The non-linearity is caused by the pinned ends not being allowed to move to any appreciable extent relative to the initial coordinates of the beam ends. It is to be noted that the axial inertia is neglected. The boundary conditions and the initial conditions for Eq.(1) are:

$$w(x,t) = w_{,xx}(x,t) = 0 \quad \text{at } x = 0 \text{ and } L, \tag{3}$$

$$w(L/2,0) = w_{max},\ w_{,t}(L/2,0) = 0 \tag{4}$$

It is convenient to work with dimensionless quantities, since such a formulation will facilitate the identification of the order of magnitude of some variables. Hence let

$$\hat{t} = \omega t, \quad \hat{x} = x/L, \quad \hat{w} = w/L, \tag{5}$$

where ω is a characteristic circular frequency (not yet defined).

Using the chain rule of differentiation Eq.(1) becomes

$$w_{,\hat{x}\hat{x}\hat{x}\hat{x}} - \hat{N}_x w_{,\hat{x}\hat{x}} + \frac{m L^4 \omega^2}{E I} \hat{w}_{,\hat{t}\hat{t}} = 0, \tag{6}$$

where

$$\hat{N}_x = \frac{N_0 L^2}{E I} + \frac{A L^2}{2 I} \int_0^1 \hat{w}_{,\hat{x}}^2\, d\hat{x}. \tag{7}$$

For notational convenience, one omits the caret on all the quantities and variables to get

$$w_{,xxxx} - N_x w_{,xx} + \frac{m L^4 \omega^2}{E I} w_{,tt} = 0. \tag{8}$$

The dimensionless boundary and initial conditions given by Eqs. (3) and (4) become

$$w(x,t) = w_{,xx}(x,t) = 0 \qquad \text{at } x = 0 \text{ and } 1, \tag{9}$$

$$w(1/2, 0) = w_{max}/L, \qquad w_{,t}(1/2, 0) = 0. \tag{10}$$

To get Eq. (9) into a Duffing equation form, one lets

$$w(x,t) = \phi(x)\, q(t) \tag{11}$$

where $\phi(x)$ is the characteristic mode of a simply supported beam:

$$\phi(x) = \sin(n\pi x); \qquad n = 1,2,3 \ldots \tag{12}$$

Substituting Eqs. (11) and (12) into Eq. (8) and performing the integration given by Eq. (7) one obtains

$$\ddot{q} + \frac{E I}{m L^4 \omega^2} \left(n^4 \pi^4 + \frac{N_0 L^2 n^2 \pi^2}{EL}\right) q + \frac{n^4 \pi^4 A E}{4 m \omega^2 L^2} q^3 = 0. \tag{13}$$

If one choose ω to be

$$\omega = \left[\frac{E I}{m L^4} \left(n^4 \pi^4 + \frac{N_0 L^2 n^2 \pi^2}{E I}\right) \right]^{1/2} \tag{14}$$

Then Eq. (13) becomes

$$\ddot{q} + q + M q^3 = 0, \tag{15}$$

where the dimensionless parameter M is given by

$$M = \frac{n^4 \pi^4 A L^2}{4 I} \Big/ \left(n^4 \pi^4 + N_0 L^2 n^2 \pi^2 / E I\right). \tag{16}$$

It is to be noted that for a square cross section, $I = bh^3/12$, and for N_0, the value of M is $3L^2/h^2$ which is a big number. To introduce a small parameter in Eq. (15), one lets

$$q = \varepsilon u, \qquad \varepsilon = 1/M. \tag{17}$$

Then Eq. (15) becomes

$$\ddot{u} + u + \varepsilon u^3 = 0, \qquad \text{where } \varepsilon << 1. \tag{18}$$

Equation (18) is a Duffing-type equation which represents a non-linear oscillator without damping [16]. It is to be noted that the non-linearity is of the hardening type.

The initial conditions given by Eq. (14) become

$$u(0) = w_{max} / \varepsilon L, \qquad \dot{u}(0) = 0. \tag{19}$$

Application of the Method of Multiple Scales

A number of perturbation methods are applicable in seeking approximate solution to Eq. (18). These include the method of harmonic balance, Lindstedt-Poincare', equivalent linearization, Krylov-Bogoliubv-Mitropolski, generalized averaging, Lie series and transformations, and multiple scales method. In the former three methods one seeks a periodic solution which is assumed a priori to occur. The later four methods yield a set of first order differential equations which describe the slow time evolution of the amplitude and phase of the response. These methods have been described thoroughly by Nayfeh [17] and Nayfeh and Mook [18].

The present work is concerned with illustrating the use of multiple scales method to construct a uniformly valid third order solution to Eq. (18). A new time $\tau = \Omega t$ is first defined, where Ω is the actual frequency, transforming Eq. (18) to

$$\Omega^2 u'' + u + \varepsilon u^3 = 0, \tag{20}$$

where the prime indicates the derivative with respect to τ. One notes that the actual frequency of the system now appears explicitly in the equation. Both Ω and u are unknowns. One seeks approximate solution for them in a form of power series in terms of ε. To do this, we assume the following forms for $u(\tau)$ and Ω^2

$$u(\tau) = u_0(\tau) + \varepsilon u_1(\tau) + \varepsilon^2 u_2(\tau) + \varepsilon^3 u_3(\tau) + \ldots, \tag{21}$$

$$\Omega^2 = 1 + \varepsilon\Omega_1 + \varepsilon^2 \Omega_2 + \varepsilon^3 \Omega_3 + \ldots . \tag{22}$$

One notes that the first term in the expansion of Ω^2 is the linear frequency which is unity. The corrections to the linear frequency are determined in the course of analysis by requiring the expansion for u to be uniform for all values of τ.

To this end, one defines a fast time scale $T_0 = \tau$, on which the main oscillatory behavior occurs, and slow time scales $T_n = \varepsilon^n \tau$, $n \geqslant 1$, on which amplitude and phase modulation takes place. The operation of time differentiation is thus expanded as

$$\frac{d}{d\tau} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \varepsilon^3 D_3 + O(\varepsilon^4), \tag{23}$$

$$\frac{d^2}{d\tau^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 (D_1^2 + 2 D_0 D_2) + 2\varepsilon^3 (D_1 D_2 + D_0 D_3) + O(\varepsilon^4), \tag{24}$$

where $D_n = \dfrac{\partial}{\partial T_n}$

We next substitute Eqs. (21-24) into Eq. (20) equate the like powers of ε to zero to obtain a hierarchy of linear differential equations which are to be solved successively. These equations are:

$$D_0^2 u_0 + u_0 = 0, \tag{25}$$

$$D_0^2 u_1 + u_1 = -2D_1D_0u_0 - \Omega_1 D_0^2 u_0 - u_0^3, \tag{26}$$

$$\begin{aligned} D_0^2 u_2 + u_2 = &- \Omega_2 D_0^2 u_0 - (D_1^2 + 2 D_2 D_0) u_0 - 2\Omega_1 D_1 D_0 u_0 \\ &- 2 D_1 D_0 u_1 - \Omega_1 D_0^2 u_1 - 3 u_0^2 u_1 \end{aligned} \tag{27}$$

$$\begin{aligned} D_0^2 u_3 + u_3 = &- 2(D_3 D_0 + D_2 D_1) u_0 - \Omega_1 (D_1^2 + 2 D_2 D_0) u_0 \\ &- 2\Omega_2 D_1 D_0 u_0 - \Omega_3 D_0^2 u_0 - (D_1^2 + 2 D_2 D_0) u_1 \\ &- 2\Omega_1 D_1 D_0 u_1 - \Omega_2 D_0^2 u_1 - 2 D_1 D_0 u_2 - \Omega_1 D_0^2 u_2 \\ &- 3 u_0^2 u_2 - 3 u_0 u_1^2. \end{aligned} \tag{28}$$

The solution of Eq. (25) is

$$u_0\,(T_0, T_1, T_2, T_3, \ldots) = A\,(T_1, T_2, T_3, \ldots)\,e^{iT_0} + cc, \tag{29}$$

where cc stands for the complex conjugate of the preceding terms and the notation indicates that only the scales T_0, T_1, T_2 and T_3 are used because the solution is to be obtained to the third order. The complex amplitude $A(T_1, T_2, T_3, \ldots) = \frac{1}{2}\,a\,(T_1, T_2, T_3, \ldots)\exp[i\,\phi\,(T_1, T_2, T_3, \ldots)]$ is a function of the slow scales with both the amplitude a and the phase ϕ real. The use of Eq. (29) in Eq. (26) then yields the following inhomogenous equation for $u_1\,(T_0, T_1, T_2, T_3, \ldots)$:

$$D_0^2\,u_1 + u_1 = (\Omega_1\,A - 2iD_1\,A - 3\,A^2\,\overline{A})\,e^{iT_0} - A^3\,e^{3iT_0} + cc, \tag{30}$$

where $\overline{A}$ is the complex conjugate of A.

The first term on the right side of Eq. (30) is resonant, and one requires it vanish in order to avoid secular behavior in u_1. Thus one obtains

$$2iD_1\,A = \Omega_1\,A - 3\,A^2\,\overline{A}. \tag{31}$$

This equation defines the rate of change of A on the slow time scale T_1. According to the method of multiple scales $D_1\,A = 0$ for periodic motion. Therefore

$$\Omega_1 = 3A\,A. \tag{32a}$$

Replacing A by its polar form yields

$$\Omega_1 = \frac{3}{4}\,a^2. \tag{32b}$$

The next step one solves Eq. (30) for u_1 retaining only the particular solution to obtain

$$u_1\,(T_0, T_1, T_2, T_3, \ldots) = \frac{A^3}{8}\;e^{3iT_0} + cc. \tag{33}$$

At this end, one substitutes Eq. (29) and Eq. (33) into Eq. (27) to obtain the following equation for $u_2 (T_0, T_1, T_2, T_3, \ldots)$:

$$D_0^2 u_2 + u_2 = -(2iD_2 A + D_1^2 A + 2i\,\Omega_1 D_1 \bar{A}\,\Omega_2 A + \frac{3}{8} A^3 \bar{A}^2)\, e^{iT_0}$$
$$-(\frac{9}{4} i A^2 D_1 A + \frac{3}{4} A^4 \bar{A} - \frac{9}{8} \Omega_1 A^3)\, e^{3iT_0} - \frac{3}{8} A^5 e^{5iT_0} + cc. \quad (34)$$

The annulment of secular terms in u_2 requires that

$$2iD_2 A + D_1^2 A + 2\,i\,\Omega_1 D_1 A - \Omega_2 A + \frac{3}{8} A^3 \bar{A}^2 = 0. \quad (35)$$

Setting $D_i A \equiv 0$, $i = 1,2$, in the previous equation one obtains

$$\Omega_2 = \frac{3}{8} A^2 A^2. \quad (36a)$$

One repleaces A in Eq. (36a) by its polar form to obtain

$$\Omega_2 = \frac{3}{128} a^4. \quad (36b)$$

The solution for u_2 with only retaining the particular solution is

$$u_2 (T_0, T_1, T_2, T_3, \ldots) = -\frac{21}{64} A^4 \bar{A}\, e^{3iT_0} + \frac{1}{64} A^5 e^{5iT_0} + cc. \quad (37)$$

The governing equation for $u_3 (T_0, T_1, T_2, T_3, \ldots)$ is obtained by the substitution of Eqs. (29), (33) and (37) into Eq. (28). The result is

$$D_0^2 u_3 + u_3 = -[2iD_3 A + 2 D_2 D_1 A + \Omega_1 (D_1^2 A + 2iD_2 A) + 2iD_1 A$$
$$+ 2i\Omega_2 D_1 \bar{A} - \Omega_3 A - \frac{57}{64} A^4 \bar{A}^3]\, e^{iT_0}$$
$$-[-\frac{63}{6} iA^3 \bar{A} D_1 A - \frac{63}{32} iA^4 D_1 \bar{A} + \frac{3}{4} A (D_1 A)^2$$
$$+ \frac{3}{8} A^2 D_1^2 A + \frac{9}{4} iA^2 D_2 A + \frac{189}{64} \Omega_1 A^4 \bar{A}$$
$$+ \frac{9}{4} i\Omega_1 A^2 D_1 A - \frac{9}{8} \Omega_2 A^3 - \frac{123}{64} A^5 \bar{A}^2]\, e^{3iT_0}$$

$$+ \left[\frac{50}{64}\ i\, A^4 D_1 A - \frac{25}{64}\ \Omega_1 A^5\ - \frac{54}{64}\ A^6 \bar{A}\right] e^{5iT_0}$$

$$- \frac{3}{32}\ A^7 e^{7iT_0}\ + \text{cc}. \tag{38}$$

Eliminating the secular terms from u_3, one obtains from Eq. (38) that

$$2iD_3 A + 2\, D_2 D_1 A + \Omega_1 (D_1^2 A + 2iD_2 A) + 2iD_1 A$$

$$+ 2\, i\, \Omega_2 D_1 \bar{A} - \Omega_3 A - \frac{57}{64}\ A^4 \bar{A}^3 = 0. \tag{39}$$

Requiring that $D_i A \equiv 0, \quad i = 1,2,3$, in Eq. (39) separately vanish yields

$$\Omega_3 = - \frac{57}{64}\ A^3 \bar{A}^3. \tag{40a}$$

Upon substituting the polar form for A one obtsins

$$\Omega_3 = - \frac{57}{4096}\ a^3. \tag{40b}$$

These leave the following perturbational equation for $u_3\ (T_0, T_1, T_2, T_3, \ldots)$:

$$D_0^2 u_3 + u_3 = - \left(\frac{189}{64} \Omega_1 A^4 \bar{A} - \frac{9}{8}\ \Omega_2 A^3 - \frac{123}{64} A^5 \bar{A}^2\right) e^{3iT_0}$$

$$- \left(\frac{25}{64}\ \Omega_1 A^5 + \frac{54}{64} A^6 \bar{A}\right) e^{5iT_0} - \frac{3}{32}\ A^7 e^{7iT_0} + \text{cc}. \tag{41}$$

The solution of Eq. (41) retaining only the particular solution is

$$u_3\,(T_0, T_1, T_2, T_3, \ldots) = \frac{1}{512}\ \left(417\, A^5 \bar{A}^2 e^{3iT_0} - 43\, A^6 \bar{A}\, e^{5iT_0}\right.$$

$$\left. + A^7 e^{7iT_0}\right) + \text{cc}. \tag{42}$$

where use has been made of Eqs. (32a) and (36a).

It is to be noted that the use of the symbolic manipulation makes it relatively easy in conducting higher order calculations.

The substitution for u_0, u_1, u_2, and u_3 from Eqs. (29), (33), (37) and (42) into Eq. (21) and replacing A by its polar form and T_0 by τ gives

$$u = \frac{a}{2}\ e^{i(\tau+\phi)} + \varepsilon\ \frac{a^3}{64}\ e^{3i(\tau+\phi)} - \varepsilon^2 \frac{21\,a^5}{2048}\ [e^{3i(\tau+\phi)} - \frac{1}{21}\ e^{5i(\tau+\phi)}] + \varepsilon^3\ \frac{a^7}{65536}\ [\,417\,e^{3i(\tau+\phi)} - 43\,e^{5i(\tau+\phi)} + e^{7i(\tau+\phi)}\,] + cc. \qquad (43)$$

The frequency-amplitude relation is determined by superimposing Eqs. (32b), (36b) and (40b) according to Eq. (22). Specifically

$$\Omega^2 = 1 + \frac{3}{4}\ \varepsilon\, a^2 + \frac{3}{128}\ \varepsilon^2 a^4 - \frac{57}{4096}\ \varepsilon^3 a^6 + \ldots \qquad (44)$$

Upon taking the square root of Eq. (44) one obtsins

$$\Omega = 1 + \frac{3}{8}\ \varepsilon\, a^2 - \frac{15}{256}\ \varepsilon^2 a^4 + \frac{123}{8192}\ \varepsilon^3 a^6 + \ldots \qquad (45)$$

It is to be remarked that the first two terms in the right hand side of Eqs. (44) and (45) are the same as that given in references [8]; [9]; [10]; [12] and [13] as a first order approximation.

Results and Discussion

To compare the frequency-amplitude relation given by Eqs. (44) and (45) with that of the previous investigators, one substitutes $\varepsilon\, a^2 = \frac{1}{4}\ (\frac{w_{max}}{\rho})^2$, where $\rho\ (= \sqrt{I/A})$ is the radius of gyration of the cross section. The exact solution of Eq. (18) is an elliptic function of the first kind [16,19]. The exact frequency amplitude relation is given by

$$\Omega_{exact} = \frac{\pi}{2}\ (1 + \varepsilon\, a^2\,)^{1/2}\, K(k), \qquad (46a)$$

where K(k) is the complete elliptic integral of the first kind

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{(1 - k^2 \sin^2\theta)^{1/2}} , \quad k^2 = \frac{\varepsilon a^2}{2(1 + \varepsilon a^2)} \tag{46b}$$

The integration given by Eq. (46b) is a regular integral which can be computed easily. It is also tabulated [20]. In addition, one can readily obtain an approximate value for the frequency given by Eqs. (46a) by expanding the integrand in Eq. (46b) and integrating term by term, that is

$$K(k) = \int_0^{\pi/2} \left(1 + \frac{1}{2} k^2 \sin^2\theta + \frac{3}{8} k^4 \sin^4\theta + \frac{5}{16} k^6 \sin^6\theta + \ldots\right)$$

$$= \frac{\pi}{2} \left(1 + \frac{1}{4} k^2 + \frac{9}{64} k^4 + \frac{81}{256} k^6 + \ldots\right). \tag{47}$$

The substitution of Eq. (47) into Eq. 46a) yields

$$\Omega_a = 1 + \frac{3}{8} \varepsilon a^2 - \frac{21}{256} \varepsilon^2 a^4 + \frac{81}{2048} \varepsilon^3 a^6 + \ldots . \tag{48}$$

The non-linear frequency Ω for specified values of the maximum amplitude ratio (w_{max}/ρ) have been calculated and presented in Table 1. The present solution is in excellent agreement with the exact solution obtained from direct integration of the elliptic function. For small maximum amplitude ratio, equivalently small εa^2, the corresponding frequencies agree with the exact values up to the fourth decimal place. As execpted, for large maximum amplitude ratio, one needs to include more terms in the perturbation series to obtain more accurate results.

Figure 1 displays the linear and the non-linear displacements u over the fundamental period for $a = 2.0$, $\varepsilon = 0.1$, and correspondingly $w_{max}/\rho = 1.2649$ and $\Omega = 1.1414$. The first few harmonic amplitudes C_n, where $u = \sum C_n \cos(n\tau + \phi_n)$, are small compared to the fundamental. These values are $C_1 = 2$, $C_3 = 2.0066 \times 10^{-2}$, $C_5 = 1.4453 \times 10^{-4}$, $C_7 = 3.9062 \times 10^{-5}$. It is clear from Eq. (43) that only odd harmonics exist. Shown in the same Figure are the non-linear displacements for $\varepsilon = 0.2$ ($\Omega =$

Table 1. Non-linear frequency Ω at various amplitudes (w_{max}/ρ); comparison of solutions obtained by other methods (ρ = radius of gyration)

$\frac{W_{max}}{\rho}$	Present study Eq. (44)	Elliptic function Exact Eq. (46)	Elliptic function Approx. Eq. (48)	Rayleigh-Ritz method [8]	Finite element method [9]
0.2	1.0037 (1.0037)*	1.0037	1.0037	1.0025	–
0.4	1.0149 (1.0149)	1.0149	1.0149	1.0099	1.0198
0.6	1.0333 (1.0333)	1.0331	1.0331	1.0222	–
0.8	1.0586 (1.0586)	1.0580	1.0581	1.0392	1.0770
1.0	1.0903 (1.0903)	1.0892	1.0892	1.0606	1.1180
2.0	1.3265 (1.3314)	1.3178	1.3325	1.2247	1.4142
3.0	1.6272 (1.7181)	1.6257	1.8790	1.4577	1.8028

* Values in parenthesis are from Eq. (45)

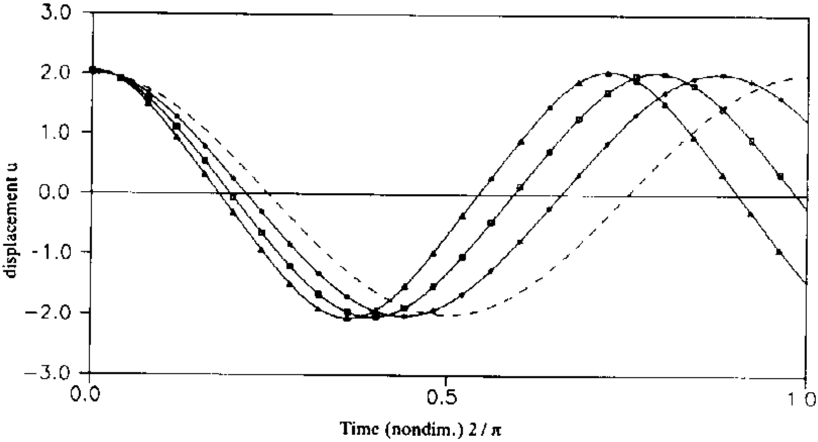

Fig. 1. Transverse displacements for a = 2.

1.2680, $w_{max}/\rho = 1.7889$) and $\varepsilon = 0.3$ ($\Omega = 1.3829$, $w_{max}/\rho = 2.1909$). The corresponding sets of harmonic amplitudes are $C_1 = 2$, $C_3 = 3.6781 \times 10^{-2}$, $C_5 = 9.3740 \times 10^{-5}$, $C_7 = 3.1250 \times 10^{-5}$ and $C_1 = 2$, $C_3 = 5.9917 \times 10^{-2}$, $C_5 = 1.7227 \times 10^{-3}$, $C_7 = 1.0547 \times 10^{-4}$, respectively. The harmonic amplitudes increase with increasing ε, however they are still small compared to the fundamental.

The applicability of the method of multiple scales to study non-liear vibrations of immovable pinned ends beam has been demonstrated. Representation of the non-linear frequency and the displacement explicitly in terms of the amplitude can provide means by which the effect of a parameter change on a system can readily be gauged, which is useful in design process.

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الاهتزازات الحُرّة اللاخطيّة لكمرة مثبّتة تثبيتًا بسيطًا

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ملخص البحث. تم فحص الاهتزازات الحُرّة لكمرة مثبتة تثبيتًا بسيطًا والخاضعة لتقوّسات عرضيّة كبيرة. وقد تم تبسيط معادلة الحركة إلى شكل معادلة الدفنج ذات معامل الاضطراب الصغير. لقد استخدمت طريقة المقاييس متعدّدة الاضطراب لتحديد حل منتظم صحيح من الدرجة الثالثة. وقد كانت التوقعات للترددات اللاخطيّة في اتفاق تام مع الحل الرياضي الصحيح بطريقة التكامل المباشر لتكامل القِطْع الناقص. وأيضًا تمّ عرض محتويات الإزاحات العرضيّة اللاخطيّة الهارمونيّة.