

## **Thermal Deformation of Antisymmetric Angle-ply Laminated Plate Strips in Cylindrical Bending**

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**Abstract.** Exact analytical solutions for the thermal deformations of antisymmetric angle-ply laminated plate strips in cylindrical bending are developed. The state variable technique in conjunction with Jordan canonical form is used to obtain these solutions for plate strips with arbitrary boundary conditions. Deflections are computed for laminates subjected to linearly varying transverse temperature distribution. Numerical results are presented emphasizing the effects of shear deformation, number of layers, ply-angles, transverse shear correction factor and boundary conditions on the thermal response.

### **Introduction**

Thermal deformations and stress resultants in rectangular, antisymmetric cross-ply, and angle-ply thin laminates are investigated in [1]. Exact solutions are obtained for the response of simply supported plates to general three-dimensional temperature variations. An incremental loading procedure [2] is used to obtain approximate solutions for the moderately large deflections of antisymmetric angle-ply plates under nonuniform temperature fields. A parabolic in-plane variation and a linearly varying transverse distribution of temperature are considered. Results based on the large-deflection theory are compared with exact solutions to the classical small-deformation formulation for various fiber reinforced laminates. Analytical three-dimensional elasticity solutions are presented in [3] for the stress and free vibration problems of multilayered anisotropic plates with perfectly bonded layers. The plates are assumed to have rectangular geometry and an antisymmetric lamination with respect to the middle plane. Each of the plate stresses and displacements is decomposed into symmetric and antisymmetric components in the thickness direction, and is expressed in terms of a double Fourier series in the Cartesian surface coordinates. Extensive numerical results are presented showing the effects of variation in the lamination and geometric parameters of composite

plates on the importance of the transverse stress and strain components. Exact analytical solutions for the thermal deformations in antisymmetric angle-ply laminated plates are developed in [4]. A generalized Levy-type procedure in conjunction with the state variables technique is used to obtain these solutions for rectangular plates which are simply supported on two opposite edges and have arbitrary boundary conditions on the remaining ones. Thermal deformations in symmetric and antisymmetric cross-ply beams are investigated in [5]. Various beam theories are used in the analysis.

In the present work, the state space concept in conjunction with Jordan canonical form is used to solve exactly the thermoelastic governing equations of the first order shear deformation and classical theories of antisymmetric angle-ply laminated plate strips in cylindrical bending. The numerical applications are presented to show the influence of plate strip parameters and boundary conditions on the thermal deflection. The present paper extends the earlier work in [5] on cross-ply laminated beams.

### Governing Equations

Consider a laminated rectangular plate strip that is very long in the  $y$ -direction and has a finite dimension  $L$  along the  $x$ -direction. All displacement quantities ( $u, v, w, \varphi, \psi$ ) are functions of only  $x$ . All derivatives with respect to  $y$  are zero and the plate bends into a cylindrical surface. The resultants of the first-order shear deformation plate strip theory (FOBT) in cylindrical bending for antisymmetric angle-ply laminates are given by:

$$\begin{aligned}
 N_x &= A_{11}u' + B_{16}\varphi' - N_x^T \\
 N_y &= A_{12}u' + B_{26}\varphi' - N_y^T \\
 N_{xy} &= A_{66}v' + B_{16}\varphi' - N_{xy}^T \\
 M_x &= B_{16}v' + D_{11}\varphi' - M_x^T \\
 M_y &= B_{26}v' + D_{12}\varphi' - M_y^T \\
 M_{xy} &= B_{16}u' + D_{66}\varphi' - M_{xy}^T \\
 Q_y &= KA_{44}\varphi \\
 Q_x &= KA_{55}(w' + \varphi)
 \end{aligned} \tag{1}$$

$(N_x, N_y, N_{xy}), (M_x, M_y, M_{xy})$  are the force and moment resultants respectively;  $Q_x$  and  $Q_y$  are the transverse shear stress resultants;  $K$  is the transverse shear correction factor;  $(N_x^T, N_y^T, N_{xy}^T), (M_x^T, M_y^T, M_{xy}^T)$  are the thermal forces and thermal moments respectively. A prime denotes ordinary differentiation with respect to  $x$ . The laminate stiffnesses are given by:

$$\begin{aligned} (A_{ij}, B_{ij}, D_{ij}) &= \sum_{k=1}^N \int_{z_{k-1}}^{z_k} (1, z, z^2)(Q_{ij})_k dz \quad i, j = 1, 2, 6 \\ A_{ij} &= \sum_{k=1}^N \int_{z_{k-1}}^{z_k} (Q_{ij})_k dz \quad i, j = 4, 5 \end{aligned} \quad (2)$$

The thermal forces and thermal moments are defined as:

$$\begin{Bmatrix} N_x^T, M_x^T \\ N_y^T, M_y^T \\ N_{xy}^T, M_{xy}^T \end{Bmatrix} = \sum_{k=1}^N \int_{z_{k-1}}^{z_k} \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix}_k \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_k (1, z) \Delta T dz \quad (3)$$

Where  $(Q_{ij})_k$  are the material coefficients of the  $k$ th lamina in the laminate coordinate system and  $(\alpha_x, \alpha_y, \alpha_{xy})_k$  are the coefficients of linear thermal expansion for layer  $k$  in the laminate coordinates;  $\Delta T$  denotes the temperature rise in the laminate. The counterparts of equation (1) using the classical plate strip theory (CBT) in cylindrical bending are

$$\begin{aligned} N_x &= A_{11}u' - N_x^T \\ N_y &= A_{12}u' - N_y^T \\ N_{xy} &= A_{66}v' - B_{16}w'' - N_{xy}^T \\ M_x &= B_{16}v' - D_{11}w'' - M_x^T \\ M_y &= B_{26}v' - D_{12}w'' - M_y^T \\ M_{xy} &= B_{16}u' - M_{xy}^T \end{aligned} \quad (4)$$

Equilibrium equations for (FOBT) and (CBT) in terms of displacement quantities are:

FOBT

$$\begin{aligned}
 A_{11}u'' + B_{16}\varphi'' &= (N_X^T)' \\
 A_{66}v'' + B_{16}\varphi'' &= (N_{XY}^T)' \\
 KA_{55}(w'' + \varphi') &= 0 \\
 B_{16}v'' + D_{11}\varphi'' - KA_{55}(w' + \varphi) &= (M_X^T)' \\
 B_{16}u'' + D_{66}\varphi'' - KA_{44}\varphi &= (M_{XY}^T)'
 \end{aligned} \tag{5}$$

CBT

$$\begin{aligned}
 A_{11}u'' &= (N_X^T)' \\
 A_{66}v'' - B_{16}w''' &= (N_{XY}^T)' \\
 B_{16}v''' - D_{11}w''' &= (M_X^T)''
 \end{aligned} \tag{6}$$

The following boundary conditions for hinged (H), clamped (C) and free (F) supports are considered at the ends  $x = 0, L$ :

FOBT

$$\begin{aligned}
 H: \quad u = w = \varphi = N_{XY} = M_X &= 0 \\
 C: \quad u = v = w = \varphi = \varphi' &= 0 \\
 F: \quad N_X = N_{XY} = M_X = M_{XY} = Q_X &= 0
 \end{aligned} \tag{7}$$

CBT

$$\begin{aligned}
 H: \quad u = w = N_{XY} = M_X &= 0 \\
 C: \quad u = v = w = w' &= 0 \\
 F: \quad N_X = N_{XY} = M_X = (M_X)' &= 0
 \end{aligned} \tag{8}$$

### Exact Solution

The state space concept in conjunction with Jordan canonical form is used to determine the thermal deformation of antisymmetric angle-ply laminated plate strips in cylindrical bending. A linearly varying transverse temperature distribution

$$\Delta T = Cz \quad (9)$$

is considered. For this problem, the thermal forces  $N_x^T$ ,  $N_y^T$  and the thermal moment  $M_{xy}^T$  vanish. To solve for the thermal deflection  $w$ , the displacement equations of equilibrium for (FOBT) are reduced to:

$$\begin{aligned} \varphi'' &= c_1(w' + \varphi) + f_1(N_{xy}^T)' + f_2(M_x^T)' \\ w'' &= -\varphi' \end{aligned} \quad (10)$$

With the following associated deduced boundary conditions:

$$\begin{aligned} H: \quad \varphi' &= f_1 N_{xy}^T + f_2 M_x^T, \quad w = 0 \\ C: \quad w &= \varphi = 0 \\ F: \quad \varphi' &= f_1 N_{xy}^T + f_2 M_x^T, \quad w' + \varphi = 0 \end{aligned} \quad (11)$$

where

$$c_1 = \frac{KA_{55}A_{66}}{(A_{66}D_{11} - B_{16}^2)}, \quad f_1 = \frac{-B_{16}}{(A_{66}D_{11} - B_{16}^2)}, \quad f_2 = \frac{A_{66}}{(A_{66}D_{11} - B_{16}^2)} \quad (12)$$

In order to reduce the system of equation (10) to a state form, The components of the state vector  $\{z(x)\}$  are defined as:

$$z_1 = \varphi, \quad z_2 = \varphi', \quad z_3 = w, \quad z_4 = w' \quad (13)$$

Using equation (13), the system of equations (10) will be converted to the form

$$\{z'\} = [A]\{z\} + \{r\} \quad (14)$$

Where the matrix  $A$  and the load vector  $\{r\}$  are defined as:

$$[A] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ c_1 & 0 & 0 & c_1 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad \{r\} = \begin{Bmatrix} 0 \\ f_1(N_{xy}^T)' + f_2(M_x^T)' \\ 0 \\ 0 \end{Bmatrix} \quad (15)$$

The four eigenvalues of the matrix  $A$  are all equal to zero. Since the matrix  $A$  has an eigenvalue with multiplicity 4, the solution to equation (14) will be given in terms of Jordan Canonical form as:

$$\{z\} = [M]e^{[J]x}[M]^{-1}\{k\} + [M]e^{[J]x} \int^x e^{-[J]\eta} [M]^{-1} \{r(\eta)\} d\eta \quad (16)$$

Where  $[M]$  is the modal matrix that contains the eigenvector and generalized eigenvectors of the matrix  $A$

$$[M] = \begin{bmatrix} 0 & -1 & -1 & \frac{-1}{c_1} - 1 \\ 0 & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad (17)$$

,  $[J]$  is the Jordan matrix and  $e^{[J]x}$  is a block diagonal defined as:

$$e^{[J]x} = \begin{bmatrix} 1 & x & x^2/2 & x^3/6 \\ 0 & 1 & x & x^2/2 \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (18)$$

The constant vector  $\{k\}$  will be determined from the boundary conditions.

Using equations (13), (16), (17) and (18), the displacement quantities  $w$  and  $\phi$  and their derivatives will be given by:

$$\begin{Bmatrix} \varphi \\ \varphi' \\ w \\ w' \end{Bmatrix} = \begin{bmatrix} 1+c_1x^2/2 & x & 0 & c_1x^2/2 \\ c_1x & 1 & 0 & c_1x \\ -c_1x^3/6 & -x^2/2 & 1 & x-c_1x^3/6 \\ -c_1x^2/2 & -x & 0 & 1-c_1x^2/2 \end{bmatrix} \begin{Bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{Bmatrix} + \begin{Bmatrix} (f_1N_{xy}^T + f_2M_x^T)x \\ f_1N_{xy}^T + f_2M_x^T \\ -(f_1N_{xy}^T + f_2M_x^T)x^2/2 \\ -(f_1N_{xy}^T + f_2M_x^T)x \end{Bmatrix} \quad (19)$$

Substituting into equation (19) the desired combinations of boundary conditions at the ends  $x = 0, L$  defined in equation (11). One has to solve a linear system of four algebraic equations to find the constant vector  $\{k\}$ . The thermal deflection  $w$  for (H-H), (C-C), (C-F), (H-C) and (C-H) plate strips will be given by:

$$\begin{aligned} w_{H-H} &= f_3(Lx - x^2)/2 \\ w_{C-C} &= 0 \\ w_{C-F} &= -f_3x^2/2 \\ w_{H-C} &= -\frac{e_1c_1x^3}{6} + e_2\left(x - \frac{c_1x^3}{6}\right) - \frac{f_3x^2}{2} \\ w_{C-H} &= -\frac{e_3x^2}{2} + e_4\left(x - \frac{c_1x^3}{6}\right) - \frac{f_3x^2}{2} \end{aligned} \quad (20)$$

where

$$\begin{aligned} f_3 &= f_1N_{xy}^T + f_2M_x^T, \quad e_1 = -\frac{f_3L(1+c_1L^2/12)}{(1+c_1L^2/3)}, \\ e_2 &= \frac{-2(f_3L + (1+c_1L^2/2)e_1)}{c_1L^2}, \quad e_3 = \frac{-c_1f_3L^2}{2(1+c_1L^2/3)}, \quad e_4 = \frac{-e_3}{c_1L} \end{aligned} \quad (21)$$

A similar procedure can be followed to solve for the thermal deflection using (CBT) to find:

$$\begin{aligned}
 w_{H-H} &= f_3(Lx - x^2)/2 \\
 w_{C-C} &= 0 \\
 w_{C-F} &= -f_3x^2/2 \\
 w_{H-C} &= \frac{f_3}{2} \left( \frac{xL}{2} + \frac{x^3}{2L} - x^2 \right) \\
 w_{C-H} &= \frac{f_3}{4} \left( x^2 - \frac{x^3}{L} \right)
 \end{aligned} \tag{22}$$

### Numerical Results and Discussion

The thermal deflections of antisymmetric angle-ply laminated plate strips in cylindrical bending are calculated using equations (20) and (22). All laminae are assumed to be of the same thickness and made of the same orthotropic material having the following dimensionless properties:

$$E_1 = 25E_2, \quad G_{12} = G_{13} = 0.5E_2, \quad G_{23} = 0.2E_2, \quad \nu_{12} = 0.25, \quad \alpha_2 = 3 \alpha_1$$

where  $\alpha_1, \alpha_2$  are the coefficients of linear thermal expansion in the principal material directions. In all the calculations, the temperature variation is assumed to be linear in the thickness direction,  $\Delta T = Cz$ . The deflections in the figures are nondimensionalized as:

$$\bar{W} = \frac{w}{\alpha_1 CL^2}$$

The nondimensional thermal deflected shapes of antisymmetric angle-ply plate strips in cylindrical bending are shown in Fig. 1 for various boundary conditions. The theories (FOBT) and (CBT) give equal deflection for (H-H) and (C-F) plate strips and zero deflection for (C-C) plate strips, see equations (20) and (22).

The shearing deformation effect is pronounced only for (H-C) and (C-H) boundary conditions, see Fig. 2, and is relatively more significant when  $L/h \leq 15$ . In Fig. 2, the nondimensional thermal deflection is calculated at  $x = L/3$ .

The effect of ply angle and number of layers on the maximum nondimensional thermal deflection is displayed in Fig. 3, where two-, four- and ten- layers laminate cases having equal thickness are considered. For all boundary conditions, the ten-layers laminate gives higher absolute maximum deflection than four- and two-layers

counterparts at the peak values. The maximum thermal deflection occurs at  $x = L/2$  for (H-H) , at  $x = L$  for (C-F), at  $x = \frac{-f_3 + \sqrt{f_3^2 + 2c_1e_2(e_1 + e_2)}}{c_1(e_1 + e_2)}$  for (H-C) and at  $x = \frac{-(e_3 + f_3) + \sqrt{(e_3 + f_3)^2 + 2c_1e_4^2}}{c_1e_4}$  for (C-H) laminates using (FOBT). The (CBT) theory gives maximum thermal deflection at  $x = L/3$  for (H-C) and at  $x = 2L/3$  for (C-H) laminates.

In obtaining all of the above results related to (H-C) and (C-H) plate strips, it is assumed that the shear correction factor,  $K$  , is equal to  $5/6$  . The effect of using different values of  $K$  on the nondimensional thermal deflection is depicted in Figure 4. It reveals that the shear correction factor has little influence on the nondimensional thermal deflection of (C-H) plate strips. For (H-H) and (C-F) boundary conditions,  $K$  has no influence.

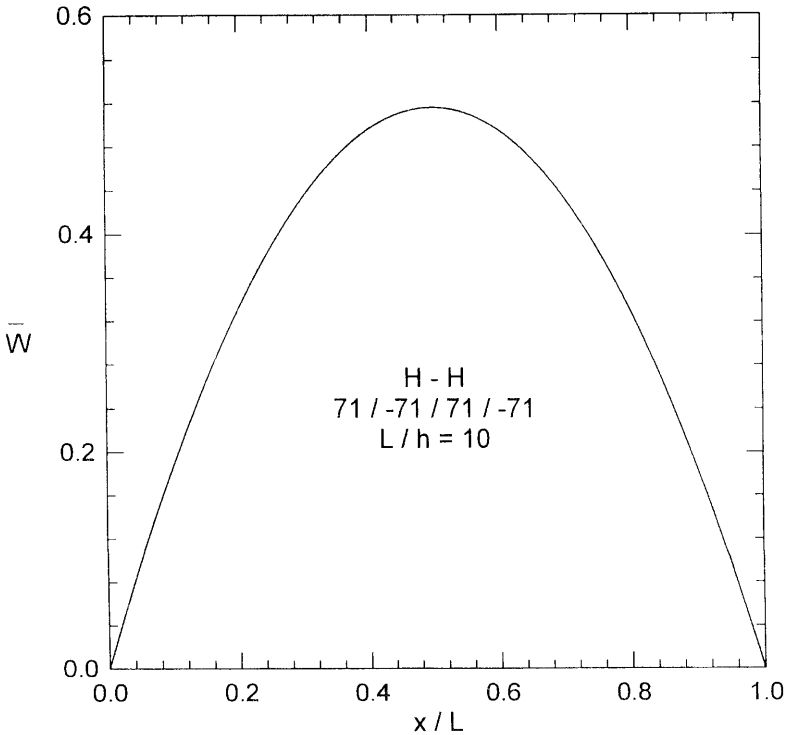


Fig. 1(a).

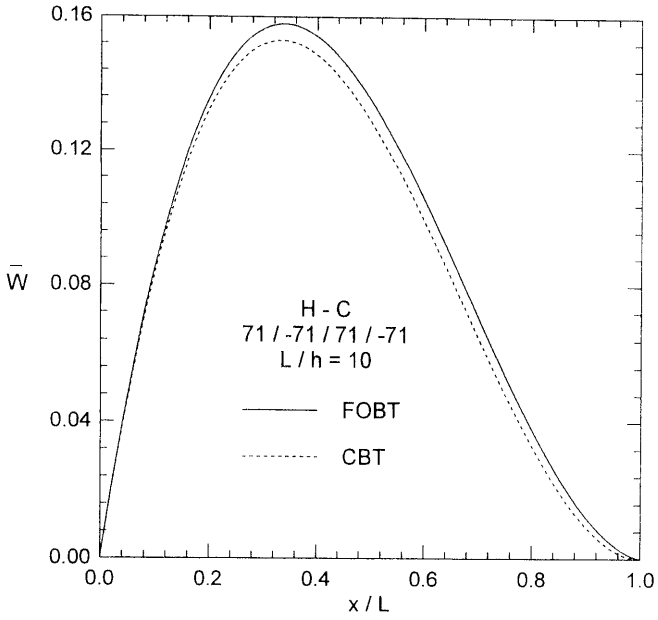


Fig. 1(b).

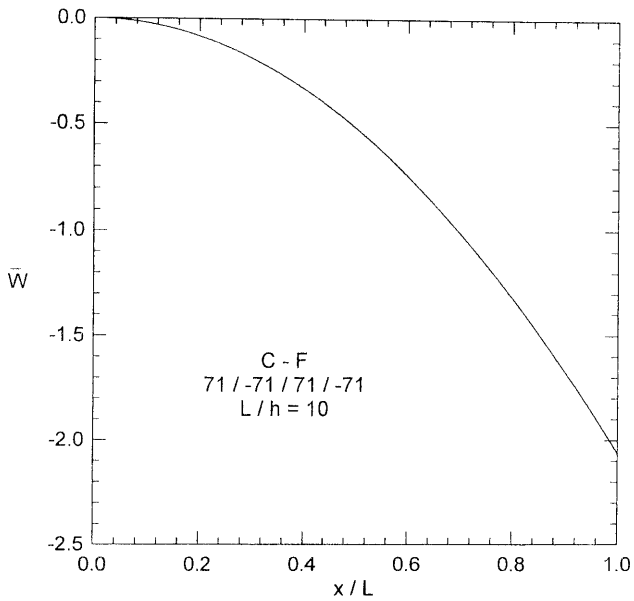


Fig. 1(c).

Fig. 1. (a-c). Nondimensional thermal deflected shape of antisymmetric angle-ply plate strip in cylindrical bending for (a) H-H, (b) H-C, (c) C-F boundary conditions.

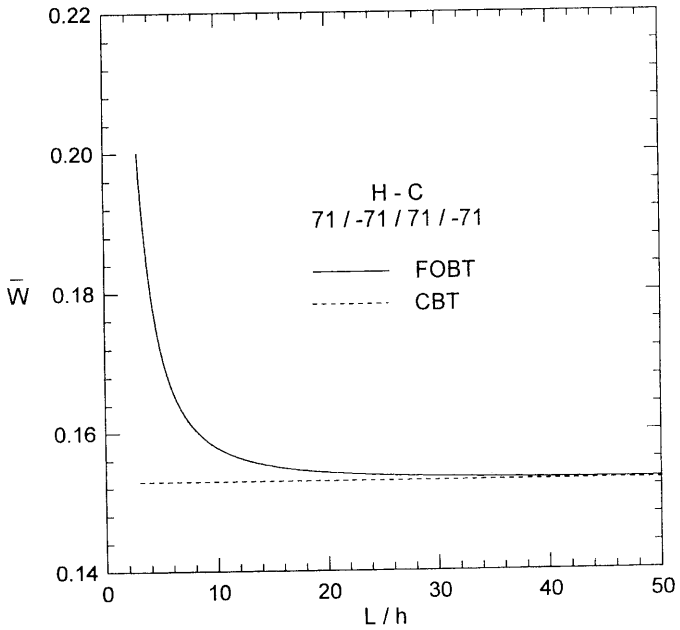


Fig. 2. Nondimensional thermal deflection vs. length to thickness ratio of H-C antisymmetric angle-ply plate strip in cylindrical bending.

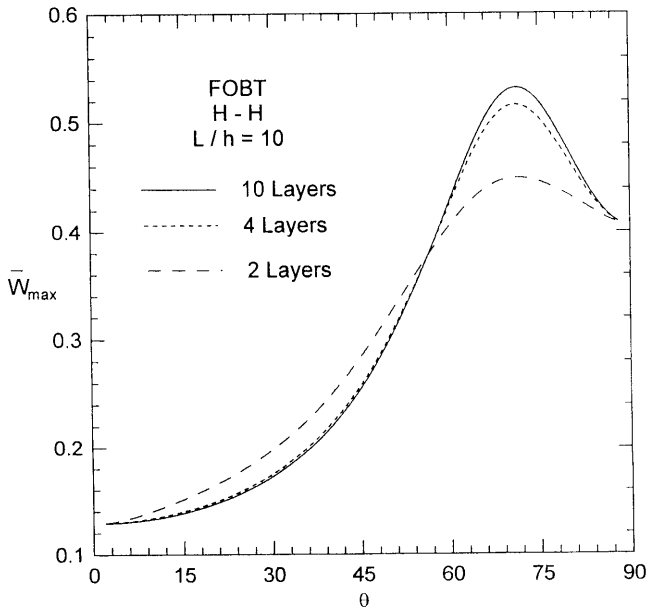


Fig. 3(a).

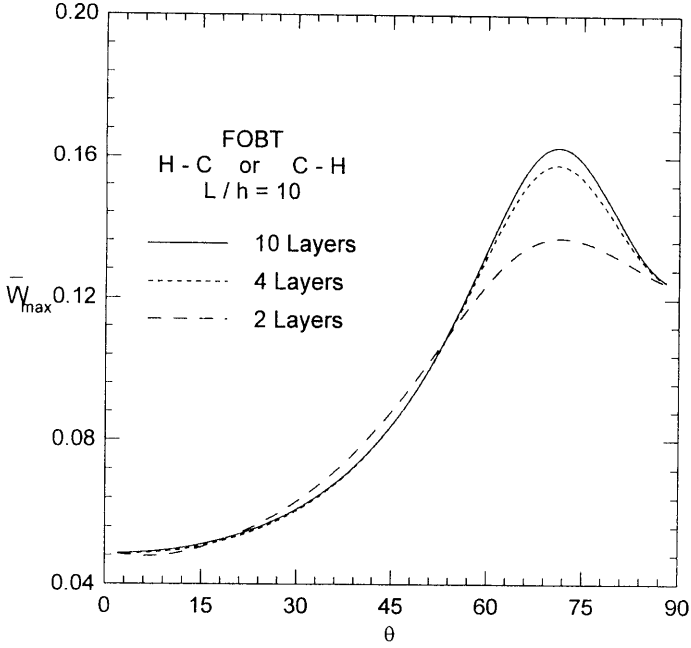


Fig. 3(b).

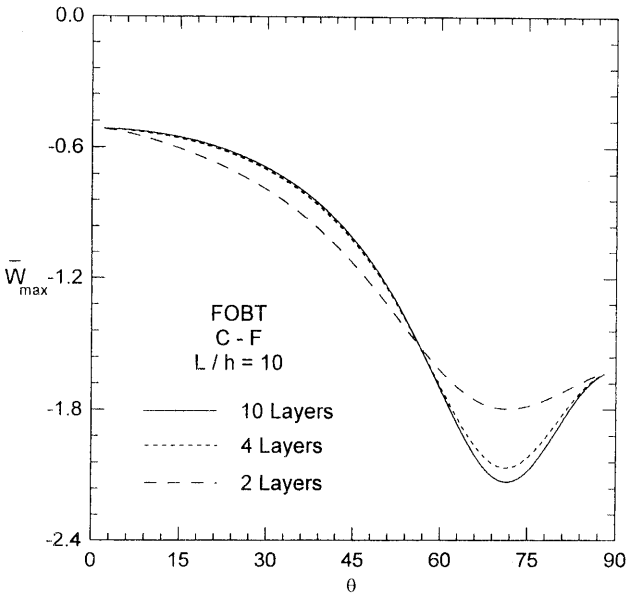


Fig. 3(c).

Fig. 3. (a-c). Effect of ply-angle and number of layers on the maximum nondimensional thermal deflection of (a) H-H, (b) H-C or C-H, (c) C-F antisymmetric angle-ply plate strip in cylindrical bending.

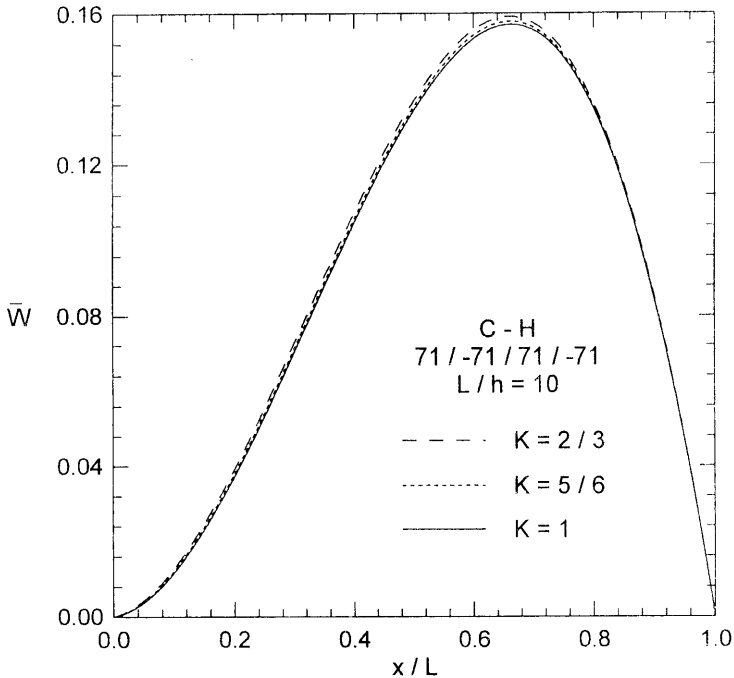


Fig. 4. Nondimensional thermal deflected shape of C-H antisymmetric angle-ply plate strip in cylindrical bending for various values of transverse shear correction factor.

### Conclusion

Thermal deformations in antisymmetric angle-ply laminated plate strips in cylindrical bending are investigated. The state space approach in conjunction with the Jordan canonical form is presented to obtain exact solutions for the thermoelastic response of antisymmetric angle-ply plate strips with arbitrary boundary conditions. The first-order shear deformation and the classical theories are used in the analysis. The two theories yield identical deflection for hinged-hinged and cantilevered plate strips and zero deflection for clamped-clamped plate strip. For clamped-hinged plate strip, the shearing deformation effect is pronounced and the first-order shear deformation theory should be used when the plate strip is thick or moderately thick. For thin laminates, either of the two theories can be used.

Concerning the mathematical tool used in the solution, (namely the state-space approach in conjunction with the Jordan canonical form), it is proven to be of great computational efficiency. It is also capable to provide solutions for various boundary conditions in a unified manner.

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## التشوّه الحراري للصفائح الشقية المنحنية أسطوانيا والمكوّنة من طبقات ذات التماثل العكسي زاويًا

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**ملخص البحث.** تم تطوير حلول صحيحة للتشوّهات الحرارية للصفائح الشقية المنحنية أسطوانيا والمكوّنة من طبقات ذات التماثل العكسي زاويًا. واستخدمت طريقة متغيرات الحالة مقرونة بقاعدة جوردان للحصول على هذه الحلول الصحيحة للصفائح الشقية ذات شروط حافة كيفية. قمنا بحساب الانحرافات لهذه الصفائح الشقية عند تعرضها لتوزيع حراري مستعرض ذا تغير خطي. تم عرض النتائج لتوضيح تأثير التشوّه القصي وعدد الطبقات والزوايا الطبقيّة ومعامل تصحيح القص المستعرض وشروط الحافة على الاستجابة الحرارية.