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Aggregation in Decision Problems: Concepts and Applications

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Abstract. This paper discusses the concept of aggregation in decision problems with the Bayesian approach. A variety of examples are provided for illustrative purposes. An aggregation error is said to occur when analyses made at aggregate and disaggregate levels yield different results. In the absence of aggregation error, perfect aggregation is said to occur. Perfect aggregation is shown to be almost impossible and consequently aggregation error is practically inevitable. Alternative measures of aggregation error are provided. Also, the impact of aggregation error on the decision to be made is analyzed.

1. Introduction

Most realistic decision problems involve a large number of variables and/or parameters. Consequently, solving such problems would be very costly, voire impossible. To reduce the size of the problems, it is common to use aggregation so that initial parameters/variables are changed into a considerably smaller number of aggregate parameters/variables. However, the solutions obtained from aggregation are frequently taken as solutions to the initial problems, neglecting aggregation errors. Unfortunately, aggregation error may be important and even misleading, a fact not widely recognized (Azaiez, 1993).

The concept of aggregation is extensively studied in a variety of fields including reliability analysis (Azaiez, 1993; Bier, 1994; Azaiez and Bier, 1994, 1995, 1996), econometrics (Theil, 1954; Simon and Ando, 1961; Ijuri, 1971; Chipman, 1975), ecological modeling (Ziegler, 1976; O'Neill and Rust, 1979; Cale and Odell, 1980; Gardens *et al.*, 1982; Cale *et al.*, 1982; Iwasa *et al.*, 1987), and automatic control theory (Sinha and Kuszta, 1976). The absence of aggregation error has been referred to in these fields variously as *total consistency* (Ijuri, 1971), *exact aggregation* (Simon and Ando, 1961), and *perfect aggregation* (Iwasa *et al.*, 1987; Azaiez and Bier, 1994, 1995, 1996). In the context of dynamic systems such as ecological models, Iwasa *et al*. (1987) defined *perfect aggregation* as follows: "If the aggregated dynamics are consistent with the original dynamics in the sense that macro-variables behave identically both in the full system and in aggregation …, then we can use the aggregated dynamics instead of the original dynamics."

Several models of aggregation of demand points in location problems have been investigated and some procedures have been suggested to predict/optimize the size of aggregation error (Rayco *et al.*, 1996; Rayco *et al.*, 1999; Francis, 1997; Andersson *et al.*, 1998).

Phenomena similar to perfect aggregation have also been encountered in decision theory. For example, identifying the conditions for perfect aggregation is similar to the problem of choosing an appropriate *small world* for analysis, as discussed by Savage, (1972). Perfect aggregation is also related to several properties involved in the combination of expert opinions. These properties include *marginalization* (McConway, 1978; Genest and Zidek, 1987; Cooke, 1991), in which the consensus probability of an event does not depend on how it is partitioned into disjoint sub-events, and *data independence* (McConway, 1978), also referred to as *external Bayesianity* (Madansky, 1978), or *prior-to-posterior coherence* (Weerahandi and Zidek, 1978), in which the consensus distribution remains unchanged regardless of whether expert opinions are combined before or after Bayesian updating.

In all this variety of fields, a common result that has been developed roughly states that aggregation error is almost inevitable. In other words, perfect aggregation is very unlikely and using aggregation as a tool for reducing the size of large problems may end up solving the wrong problem (Azaiez and Bier, 1994, 1995, 1996).

In the present study, the concept of aggregation in decision problems with Bayesian updating will be introduced. A variety of examples are discussed and results for perfect aggregation are presented. All the results in this study can be established by performing some simple analogies with other results developed by Bier (1994), and Azaiez and Bier (1994, 1995, 1996), but in a different context; namely, in Bayesian reliability analysis, and therefore proofs are omitted.

The concept is developed in Section 2. In Section 3, perfect aggregation is defined and aggregation error is discussed along with some general results for some selected measure of the error. In Section 4, results are developed for two prototype examples. Also, some interpretations and analyses are provided. In Section 5, more examples of aggregation in decision problems are discussed in order to expose the reader to a larger variety of decision problems, where this concept of aggregation arises. Finally, Section 6 serves as a conclusion.

2. The Concept of Aggregation

Most decision problems depend on uncertain factors such as demand, market conditions, interest rate, level of risk, political stability, natural disasters, etc.

The decision analyst should incorporate all parameters (deterministic and stochastic) that may considerably affect the decision in the problem formulation. In this context, assume that a random quantity Q is to be estimated in a given decision problem using Bayesian updating. Assume also that Q can be obtained from a number of random quantities $Q_1, \ldots, Q_n$. In this case, Q is called the aggregate variable and $Q_1, \ldots, Q_n$ are the disaggregate variables. Among the most common functional relationships between aggregate and disaggregate variables are the additive and multiplicative relations, given respectively by:

$$Q = \sum_{i=1}^{n} Q_i \qquad (1)$$

$$Q = \prod_{i=1}^{n} Q_i \qquad (2)$$

Let $f_i(.)$ be the prior distribution of Q_i $(1 \le i \le n)$. Also, let the set of observed data relative to Q_i be D_i. Similarly, let D be the set of aggregate data relative to quantity Q. It is frequently the case that D can be obtained from $\cup D_i$, but does not contain all the information in $\cup D_i$. For instance, if Q (respectively Q_i) designates the weekly sales of a supermarket over a country (respectively over city i $(1 \le i \le n)$), then the observations on the total weekly sales in D do not contain details on how these sales are distributed among the different cities.

In order to estimate the posterior distribution of Q from estimates of the Q_i $(1 \le i \le n)$, a range of choices is available: One approach would be to start combining prior distributions, $f_i(.)$, of the different Q_i $(i = 1, 2, \ldots, n)$ to yield a prior distribution of the quantity Q. For example, in the case of additive functional relations, this is achieved through convolution of $f_i(.)$ $(1 \le i \le n)$. Next, use the set D of observations of Q to update its prior distribution through Bayes theorem to obtain a posterior distribution of Q. Alternatively, one would start by obtaining a posterior distribution of each quantity Qi, by combining prior distribution $f_i(.)$ of each Q_i with its corresponding data set D_i. The posterior distribution of the different Q_i's are then obtained and combined using the functional relationship between Q and the Q_i's to yield a posterior distribution of Q. The former approach in which only data for Q is used is called aggregate analysis. The latter approach that exploits all information at the level of the Q_i's is called a disaggregate analysis. A disaggregate analysis that uses more information in the assessment procedure tends to be more accurate but also more costly than the aggregate analysis, as it requires more data collection and analysis. There are also other possible ways to assess Q through intermediate levels of aggregation. In these cases, instead of aggregating all the Q_i's into Q, one could aggregate subgroups of the Q_i's into some aggregate quantities and then perform a disaggregate analysis on the aggregate quantities obtained, yielding a posterior distribution of Q.

In some situations, the type of analysis to be carried out may be imposed from data consideration or some external factors (e.g., disability of the existing technology to provide detailed information), and therefore an aggregate analysis would be dictated. On the other hand, it is sometimes not possible to perform an aggregate analysis and, therefore, a disaggregate type of analysis is suggested (e.g., it is frequently impossible to identify a medical diagnostic directly and the analysis of a number of symptoms is required).

There is no reason a priori that the different types of analyses yield the same results. Actually, they provide the same results only very rarely. Moreover, the decision to be selected upon the assessment of Q may depend on the type of analysis to be performed. Consequently, it is important to know when perfect aggregation holds, and in case an aggregation error occurs, what would be the size of the discrepancy and whether it would affect the decision to be made.

3. Perfect Aggregation/Aggregation Error

Definition 3.1. Perfect aggregation holds if the posterior distributions of Q obtained respectively from aggregate and disaggregate analyses coincide.

Definition 3.2. In the absence of perfect aggregation, an aggregation error is said to occur.

Remark 3.1. It is possible to establish some analogy between the concept of aggregation for a variety of decision models and in the context of Bayesian reliability analysis (Azaiez and Bier, 1994, 1995, 1996). This analogy is based on the fact that several decision problems in group decision theory would require unanimous acceptance, which is consistent with a series system that can function only if all components must function. Others would require the acceptance of at least one decision-maker, which is consistent with a parallel system that functions if at least one component is operating. Moreover, some decision policies would require a mixture of these situations that would match combined series-parallel systems or k-out-of-n configurations in a reliability theory setting. From this analogy, one can prove that perfect aggregation is nearly impossible. More specifically, it is necessary and sufficient for perfect aggregation that all prior distributions be natural conjugate with some very restrictive conditions on the parameters. Also, for perfect aggregation not only the aggregate and disaggregate analyses must agree, but also all analyses performed at any intermediate level of aggregation. This makes perfect aggregation very stringent. This will be further discussed and illustrated later, based on two prototype examples. As perfect aggregation is practically impossible, it is important to investigate aggregation error and elaborate on the impact of aggregation error on the decision to be selected.

The decision analyst is interested in knowing whether the size of aggregation error is worth the additional cost and effort needed to perform a disaggregate analysis. Therefore, at this evaluation stage, no disaggregate data are collected. Thus, for a given set of aggregate data *AD*, a number of possible combinations of disaggregate data *DD* may correspond to *AD*. For instance, in the case of a given observation on the total sales of a supermarket for an entire country, the sum of all possible combinations of sales for individual cities coincide with the aggregate observation. Consequently, for a set of aggregate data *AD*, define the set $\Omega(AD)$ to be the corresponding set of disaggregate data. Note that the decision analyst does not know before hand which candidate *DD* among all elements of $\Omega(AD)$ will actually occur. In this study, the size of aggregation error is defined as follows:

$$L_\infty = \sup_{DD \in \Omega(AD)} \left| E(Q \mid DD) - E(Q \mid AD) \right| \qquad (3)$$

In this measure of aggregation error, $E(.)$ designates the expected value operator and the aggregate (respectively disaggregate) mean of Q is denoted by $E(Q \mid AD)$ (respectively $E(Q \mid DD)$). The notation is consistent with the infinity norm, as this selected measure evaluates the size of the largest possible absolute deviation between the aggregate and the disaggregate mean. This measure, consistent with a minimax approach, is conservative. It is mainly appropriate in situations where choosing the wrong decision due to this aggregation error may have severe consequences. It is possible to show, for a large variety of aggregation models in decision problems, that this aggregation error vanishes if and only if perfect aggregation occurs.

Let U (respectively L) be the upper (respectively lower) bound on disaggregate means over $\Omega(AD)$. Then, the following results can be proven, again by establishing some analogy with results developed in Azaiez and Bier (1996):

Proposition 3.1. If $U < +\infty$, then:

a. $L_\infty = \max\{U - E(Q \mid AD); E(Q \mid AD) - L\}$

b. $(U-L)/2 \le L_\infty \le U-L$.

The first statement of Proposition 3.1 is a consequence of the fact that the aggregate mean must fall between the lower and the upper bounds of the disaggregate mean, which in turn follows from Bayes theorem. The second statement is a direct consequence of the first one.

From Proposition 3.1, it is clear that it is sufficient to know U, L, and the aggregate mean to compute aggregation error. While the bounds U, and L can be obtained for a variety of problems using techniques such as dynamic programming (Azaiez and Bier, 1996), computing the aggregate mean is usually a tedious task. Therefore, the bounds given in Proposition 3.1.b are often useful and provide a good approximation of the actual aggregation error. Note that *U-L* is always within twice the size of aggregation error independently of the size of the problem and the complexity of the model (functional relationships between aggregate and disaggregate variables, as well as ways of collecting data). It is also important to mention that *U-L* vanishes as soon as perfect aggregation occurs.

It is possible to consider other measures of aggregation error consistent, for instance, with the expected loss or the most likelihood estimation. The choice of an appropriate measure may be influenced by considerations such as the risk attitude toward the size of the error, or the analytical tractability. The chosen measure for this study, which protects against the worst possible size of error, reflects a risk-averse behavior. However, a measure based on the expected deviation between the aggregate and desegregate means would reflect a risk-neutral behavior.

The next section will illustrate the concept and provide results, using two prototype examples.

4. Illustrations

4.1. Example 4.1

A consulting firm having an excessive demand is forced to reject a number of requests of its service. Each request submitted to the firm is examined by a group of *n* experts from different areas (*n* is a positive integer) who perform a preliminary study to determine whether or not to accept to provide the requested service. This preliminary study is free of charge, but very costly to the firm. A request rejected by any expert is automatically rejected by the firm. To reduce the cost of these preliminary studies, the top management undertakes to establish a pre-selection system that would drop directly, and without a preliminary study, all those requests with limited chances for approval by the experts. An experienced engineer is assigned the task of establishing such a pre-selection system. The engineer believes that the probability *P* of acceptance of each request should be evaluated on the basis of experience and observations on decisions made by the experts on similar consulting services previously requested. If *P* is large enough, exceeding a carefully selected value p_0, then the request is transferred to the experts for preliminary study. Otherwise, it is automatically rejected. The remaining task in this pre-selection system is to set the procedure for estimating *P*. Given that the engineer seeks to incorporate old experience with a few observations on similar requests, the Bayesian approach seems to be appropriate.

In this example, the concept of aggregation can be used to evaluate the distribution of *P*. In fact, the two types of disaggregate and aggregate analyses are respectively given below.

Disaggregate analysis: Start by constructing prior distributions for P_1, …, P_n, (probabilities of accepting the request respectively by experts *1*, …, *n*). Then, update each prior distribution with observations on previous decisions made by the corresponding expert on similar requests through Bayes theorem. Finally, combine the obtained posterior distributions of the P_i's at the expert levels to yield a posterior distribution of *P*. Note that the functional relationship between *P* and P_1, …, P_n is multiplicative as in (2). Also, note that a rejected request by a particular expert is no more examined by the remaining experts. Therefore, no information is obtained regarding the opinion of these remaining experts. This dictates the way disaggregate data is collected. More details are given below.

Aggregate analysis: Combine the prior distributions of P_1, …, P_n (using the multiplicative relationship between *P* and P_1, …, P_n) to obtain a prior distribution of *P*. Next use observations on the same set of previous decisions made but at the firm level (i.e., aggregate data) to obtain a posterior distribution of *P* through Bayes theorem.

Note that the information used to obtain the aggregate posterior distribution does not specify who, among the experts, rejected a given request. It is clear that there is no guarantee that the aggregate and disaggregate posterior distributions of *P* will agree (i.e., that perfect aggregation will occur). Consequently, by using the simpler type of analysis (the aggregate analysis), it is possible to eliminate a request (through the pre-selection system) that is likely to be accepted and to generate returns to the firm. Also, it is possible to transfer a request to the experts, that is unlikely to be accepted increasing unnecessarily the preliminary study costs. The above two situations will occur in cases where the aggregate mean of *P* is below (respectively above) the value p_0, while the disaggregate mean of *P* is above (respectively below) p_0. The following result shows that perfect aggregation is very stringent (a proof of an analogous result is given in Azaiez and Bier (1994), in a context of reliability analysis of a parallel system).

4.2. Theorem 4.1

Perfect aggregation holds if and only if the prior distributions of P_i's satisfy: $P_i \sim Beta\ (c_i, d_i)$, $i = 1, 2, \ldots, n$, with $c_i = c_{i+1} + d_{i+1}$, for $i = 1, 2, \ldots, n-1$.

While conjugate prior distributions are often used, the conditions on the parameters given in Theorem 4.1 are extremely restrictive, implying that perfect aggregation is nearly impossible.

The model can be generalized to a situation where a subgroup of experts may be used in different areas instead of only one expert. In this case, a request is rejected by a given subgroup if rejected by all experts of the subgroup. To illustrate how the result on perfect aggregation is extended to this case, assume that for the first $n-1$ areas, one expert is used, while area n has a number $k(n)$ of experts that share the decision on a given request. If expert h of area n has a probability $P_{n,h}$ to accept a given request, then the probability P_n for the request to be accepted by the subgroup of experts in area n is given by:

$$P_n = 1 - \prod_{h=1}^{k(n)} (1 - P_{n,h}) \qquad (4)$$

The following result specifies the conditions for perfect aggregation in this modified version of the problem (a proof of a more general analogous result is given in Azaiez and Bier (1995) in a reliability context of a series-parallel system):

4.3. Theorem 4.2

Perfect aggregation occurs if and only if the prior distributions of the different probabilities satisfy:

$P_i \sim Beta\ (c_i, d_i)$, $i = 1, 2, \ldots, n$, with $c_i = c_{i+1} + d_{i+1}$, for $i = 1, 2, \ldots, n-1$,
$P_{n,h} \sim Beta\ (c_{n,h}, d_{n,h})$, $h = 1, 2, \ldots, k(n)$,

with

$$d_{n,h} = c_{n,h+1} + d_{n,h+1},\ c_n = \sum_{h=1}^{k(n)} c_{n,h},\ d_n = d_{n,k(n)}.$$

Theorem 4.2 shows once more that perfect aggregation is very restrictive and serves as a special case of the generalization of Theorem 4.1 to more complex structures of decision policies. Both results given above are far from being intuitive. Further, the proof relies on a more surprising result that states mainly that *for the aggregate and disaggregate posterior distributions to agree (i.e., for perfect aggregation to occur) it is sufficient that their respective expected values agree.*

In Example 4.1, assume that the consulting firm receives a total of k_0 requests of its service. All requests are first transferred to expert *1* for the preliminary study. Assume that only k_1 requests are accepted ($k_1 \le k_0$) and transferred to expert *2*. Assume that expert *2* accepts k_2 requests that are transferred to expert *3* ($k_2 \le k_1$), and so on, until expert n receives say k_{n-1} requests and accepts k_n requests ($k_n \le k_{n-1}$). Therefore, the firm has received a total of k_0 requests of its service among which k_n requests are accepted. In this case, $DD = \{k_0, k_1, \ldots, k_n\}$, $AD = \{k_0, k_n\}$. Also, for fixed k_0 and k_n $\Omega\ (AD)$ is given by:

$$\Omega\,(AD) = \{\ k_0,\ k_1,\ \ldots,\ k_{n-1},\ k_n \ \big|\ \ k_n \le k_{n-1},\ \ldots,\ \le k_1 \le k_0\}$$

Now, to discuss the impact of aggregation error, assume that four experts are concerned with the preliminary study ($n = 4$). Assume also that for a category of requests, the probabilities of acceptance by each expert have prior distributions respectively given by:

$P_1 \sim Beta\ (4, 2)$; $P_2 \sim Beta\ (5, 1)$; $P_3 \sim Beta\ (1, 1)$; $P_4 \sim Beta\ (2, 2)$;

In this case, $E\ (P_1) = 2/3$, $E\ (P_2) = 5/6$, $E\ (P_3) = ½$, and $E\ (P_4) = ½$. From Theorem 4.1, perfect aggregation does not hold as the parameters of the prior distributions do not satisfy the relationships given in the theorem. Therefore, aggregation error occurs. Now, assume that for an observation of six requests of the same category, the experts accept only two. By using dynamic programming, L and U are found to be respectively 19.1% and 26.7%. Also, aggregation error is found to be 4.3%. Here, the upper bound U – L is nearly twice as much as aggregation error (which is almost the worst that can happen). The aggregate mean of the probability of acceptance of such requests is found to be 22.4%. Therefore, aggregation error is within 20% of the aggregate mean. As a consequence, an aggregate analysis seems to be reasonable. It is important to notice that if the value of p_0 is sufficiently large (40% say), then both aggregate and disaggregate analyses will yield the same decision of rejecting any request of that category. Assume, however, that p_0 is only 25% (a more conservative value selected to prevent from rejecting requests that are likely to be accepted by the experts), then an aggregate analysis would recommend rejecting any request of that category. This is justified from the fact that the aggregate mean is lower than p_0. On the other hand, a disaggregate analysis may recommend transferring the request to the experts as the disaggregate mean may reach as much as 26.7% exceeding p_0. Thus, the decision to be made may depend on the type of analysis to be performed.

Next, another example of decision problem involving the concept of aggregation is presented.

4.4. Example 4.2

The consulting firm of Example 4.1 was facing the risk of paying huge penalties for delay as a result of loading its experts with the preliminary studies. The top management was planning to recruit more experts to avoid penalties. However, with the important load reduction of the experts as a result of installing the pre-selection system, the recruitment decision needs to be revised. After a first investigation, it has been determined that the recruitment is still needed if the new rate of request arrivals to the experts is above a certain pre-assigned value μ_0. Otherwise, the current team of experts can manage to accomplish the consulting tasks on time. A Poisson process with an unknown frequency λ is found to be a good approximation of request arrivals to the firm. Each request has a probability π of passing the pre-selection test and to be transferred to the experts. (Note that $\pi = Pr\ [P > p_0]$). Consequently, the request arrivals to the experts follow a Poisson process with frequency $\mu = \lambda\pi$. Therefore, the decision criterion is to recruit more experts only if $\mu > \mu_0$.

The concept of aggregation is involved in estimating μ.

Disaggregate analysis: Start by constructing prior distributions for λ and π. Then, collect observations on request arrivals to the firm to compute the posterior distribution of λ. Also, collect observations on results of the pre-selection system on similar requests and compute the posterior distribution of π. Finally, using the fact that $\mu = \lambda\pi$, compute the posterior distribution of μ through Bayes theorem.

Aggregate analysis: Start by constructing prior distributions for λ and π. Next, using the fact that $\mu = \lambda\pi$, compute the prior distribution of μ. Then, collect observations on the arrivals of similar requests to the experts. Finally, compute the posterior distribution of μ through Bayes theorem.

Note that the aggregate analysis neglects the information on the request arrivals to the firm and thus the rejection rate at the pre-selection system. The conditions for perfect aggregation are given by:

4.5. Theorem 4.3

Perfect aggregation holds if and only if the prior distributions of λ and π satisfy: λ ~ Gamma (a, b); π ~ Beta (c, d), with a = c + d.

Again the proof relies on the analogy with a constant failure rate system that fails after some initiating event occurs with some probability (consisting with separating Poisson processes).

Once more, perfect aggregation is found to be very stringent and an aggregate analysis would inevitably yield an aggregation error. Such an error may result in recruiting unnecessarily more experts which is a costly investment, or it may result in failing to avoid huge penalties for delay by choosing not to reinforce the team of experts that are too loaded with the preliminary studies.

Assume now that during t time units of observation k_0 requests are received by the firm and transferred to the pre-selection system. Assume also that k requests ($k \le k_0$) pass the pre-selection test and are transferred to the experts for the preliminary study. Then, $DD = \{t, k_0, k\}$ and $AD = \{t, k\}$. Also, for a fixed t and k Ω(AD) is given by:

$$\Omega\,(AD) = \{\ t,\ k_0,\ k \mid \ k \le k_0\}$$

Assume that for a period of *6* weeks of observations ($t = 6$), *4* requests are accepted by the firm ($k = 4$). Therefore, at least a total of *4* requests have arrived to the firm. Assume that the correct number is *7* ($k_0 = 7$). Assume also that:

λ ~ Gamma (6, 2); π ~ Beta (1, 2).

Therefore, the mean arrival rate of requests to the firm is 3 per week. The average chance of a request to pass the pre-selection test is 1/3, and the average arrival rate of requests to the experts is 1 request per week. From Theorem 4.3, it is clear that perfect aggregation does not hold. It is easy to show that the disaggregate mean of $\mu = \lambda\pi$ is given by:

$$E(\mu \mid k, k_0, t) = \frac{(a+k_0)(c+k)}{c+d+k_0}\,\frac{b}{1+bt}$$

where λ ~ Gamma (a, b) and π ~ Beta (c, d). Therefore, the disaggregate arrival average rate of requests to the experts in this case is also 1. The aggregate mean, however, cannot be obtained in a closed form expression and a rough estimation by *(U+L)/2* will be used. It can be shown that for $a >$

c+d (which is the case here):

$L = (c+k)b/(1+bt)$
$U = (a+k)(c+k)b/(c+k+d)(1+bt)$

In case $a < c+d$, then *L* and *U* can be obtained simply by interchanging the expressions above.

It follows that $L = 0.77$, and $U = 1.09$. A rough estimate of the aggregate mean is then:

$(U+L)/2 = 0.93$

Aggregation error lies in the interval [0.16, 0.32]. Thus, this error can be as large as 1/3 of the estimate of the aggregate mean implying that the aggregate analysis need not be accurate. The table below provides details on the sensitivity of the size of aggregation error to the variation in the parameters *a*, *b* and *c*, *d* of the *Gamma* and the *Beta* prior distributions (respectively). However, since perfect aggregation occurs if $a = c+d$, then only c and *d* will vary while *a*, *b*, *t*, and *k* will be fixed at their initial values.

Table 1. Sensitivity of aggregation error to the variation of the parameters

c	d	L	U	U-L	(U+L)/2	2(U-L)/(U+L)
0.5	0.5	0.77	1.54	0.77	1.16	0.66
1.0	3.0	1.23	1.54	0.31	1.39	0.22
0.5	5.0	0.69	0.73	0.04	0.71	0.06
1.0	5.0	1.54	1.54	0	1.54	0
1.0	9.0	1.54	2.15	0.61	1.85	0.33
1.0	99.0	1.54	16.0	14.46	8.77	1.65

The last column of Table 1 expresses the largest possible size of aggregation error as a fraction of the aggregate mean. It is clear that this size may be very significant in case the sum *c+d* (of the *Beta* conjugate prior of π) is significantly different from *a* (the first parameter of the *Gamma* conjugate prior of λ). An extreme case yields an error that is 165% larger than the aggregate mean. Also, for some other cases, the error may reach 33% and even 66% of the estimate of the aggregate mean, which is significant. Note also that when *c+d* approaches *a*, aggregation error tends to vanish yielding perfect aggregation.

The two examples above show in particular that aggregation error may be significant and possibly lead to taking the "wrong" decision.

In the next section, more examples are offered in order to expose the reader to a larger variety of decision problems where this concept of aggregation is involved. However, no analysis will be presented.

5. Other Examples

5.1. Example 5.1

In a given region, water authorities are interested in determining an optimal operating policy for a conjunctive use of ground and surface water for a planning horizon of several years. Decisions concern the total release of water to perform from the surface source, the total size of groundwater to pump from the aquifer, and possibly the total amount of artificial recharge to inject in the aquifer from the surplus, if any, of surface water. These decisions are to be made at each irrigation period of each year of the planning horizon. A fundamental stochastic factor of the decision process is the total demand of water at each irrigation period. In order to estimate this demand:

Aggregate analysis: First, estimate the distribution of this demand by using some appropriate statistical prior distribution such as the normal or the lognormal distribution. Next, update this prior distribution using the most recent observations of the demand in order to obtain a posterior distribution through Bayes theorem.

Disaggregate analysis: Use the same approach but at the level of each grower. Next, use convolution of the obtained posterior distributions to generate a posterior distribution of the total water demand in the region for a particular irrigation period.

Clearly both approaches may yield different results, and hence the "optimal" decision policy may depend on the type of analysis to be carried out in estimating water demand.

5.2. Example 5.2

An industrial firm is interested in providing a five-year warranty to its product. This initiative would be worthwhile if the reliability of the product over five years of operation is at least 95%. To estimate the reliability of the product using the Bayesian updating, one may select the type of analysis to perform.

Aggregate analysis: First, construct a prior distribution of the failure rate of the product. Then, use observed data on the performance of the product to obtain the posterior distribution of the failure rate. Finally, deduce from the failure rate the reliability of the product over five years of operation.

Disaggregate analysis: First, construct a prior distribution of the failure rate of each component of the product. Then, use observed data on the performance of each component to obtain the posterior distribution of the failure rate of every component. Next, propagate the obtained posterior distributions at the component level to the entire product (using for instance a reliability block diagram or a fault tree representation of the failure of the product) to obtain a posterior distribution of the failure rate of the product. Finally, deduce the reliability of the product over five years of operation. (For a detailed study of this example, see Azaiez and Bier (1994, 1995, 1996)).

5.3. Example 5.3

In an evaluation process of the quality of its service, a bank undertakes to estimate the average waiting time of a customer from arrival to the bank until service completion. Assume that the Bayesian updating is to be used. Once again, the concept of aggregation is involved.

Aggregate analysis: Construct a prior distribution of the total time spent by a customer in the bank from arrival to departure. Then using some observations, compute the posterior distribution of the total waiting time (both in queue and with the server) of the customer in the bank.

Disaggregate analysis: Construct a prior distribution on the waiting time in queue. Then, update the prior distribution using few observations to obtain a posterior distribution of the waiting time in queue. Similarly, construct a prior distribution for the service time and update it with observations to obtain the posterior distribution of the time spent by a customer in service. Finally, convolute the two posterior distributions to obtain a posterior distribution on the total time spent by a customer in the bank.

This example may be extended to more complex structures of queuing systems.

Hence, this concept of aggregation can be involved in a large variety of decision problems. Therefore, caution is to be taken whenever an aggregate analysis is considered since perfect aggregation is very unlikely.

6. Conclusion

The present study introduces the concept of aggregation in decision problems in a Bayesian context. This concept, which is neglected, may arise in a large variety of problems. The decision to be made may depend on the type of analysis selected. An aggregate analysis is usually practical both in terms of the size of the problem and of the cost of data collection and analysis. A disaggregate analysis that exploits all the data available is costly but usually more accurate. However, aggregation error (obtained by performing an aggregate analysis) may be significant and may result in choosing the "wrong decision". Several examples involving this concept of aggregation are discussed and some important results are given. In addition, some tools for estimating the size of the error (or its bounds) are provided. Future work will identify the optimal intermediate level of aggregation to tradeoff the cost of a type of analysis against the accuracy of its results.

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التراكمية   مشكلات ا اذ القرار:
ا فهوم والتطبيقات

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ملخص البحث. يتطرق هذا البحث إلى اعتماد التراكمية في مشكلات اتخاذ القرار باستخدام الطريقة البازية، ويستعرض البحث مجموعة من الأمثلة التوضيحية. يشار إلى الخطأ في التراكمية في صورة عدم تطابق نتائج التحاليل على المستوى التراكمي مع تلك التي يتم القيام بها بشكل مفصل، كما يشار إلى غياب الخطأ في التراكمية بالتراكمية المثلى والتي لا تحصل إلا نادرًا، بحيث أن الخطأ في التراكمية لا يكاد يمكن تفاديه. يقترح البحث مجموعة من القياسات لتقييم الخطأ في التراكمية، كما يحلل أثر الخطأ في التراكمية على القرارات التي يتم اتخاذها.