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On the Boundedness of Solutions of a Certain Differential Equation of the Fourth Order

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Abstract. This work discusses solutions and sufficient conditions for the boundedness of the solutions of the class of nonlinear differential equation of the fourth order. The equation of the form:

$$x^{(4)} + f(\dot{x},\ddot{x},\dddot{x})\ddot{x} + h(\ddot{x}) + g(\dot{x}) + \alpha_4 x = p(t)$$

constitutes the base of our work, and we give a theorem dealing with sufficient conditions for the boundedness of solutions for the above equation. In course of the proof, a function V(x,y,z,w) will play an essential role. Two lemmas are given on the function V and its time derivative $\dot{V}$, and by virtue of them the proof of the theorem is completed.

Introduction

We consider the following real non-linear differential equation of the fourth order

$$x^{(4)} + f(\dot{x},\ddot{x},\dddot{x})\dddot{x} + h(\ddot{x}) + g(\dot{x}) + \alpha_4 x = p(t)$$

in which α_4 is constant, and f,g,h, and p depend only on the arguments displayed, with the dots denoting, as usual, differentiation with respect to t. The forcing term p(t) is assumed to be continuous. The non-linear functions f,g, and h are continuous and constructed in a way that the uniqueness theorem is warranted. Moreover, we assume that the derivatives $f_{\dot{x}} \equiv \frac{\partial f}{\partial \dot{x}}$, $f_{\dddot{x}} \equiv \frac{\partial f}{\partial \dddot{x}}$ and $g'(\dot{x}) \equiv \frac{dg}{d\dot{x}}$ exists and are continuous.

Using $y = \dot{x}$, $z = \dot{y}$ and $w = \dot{z}$ the differential equation(1) can be transformed to the equivalent system

$$\dot{x} = y,\ \dot{y} = z,\ \dot{z} = w,\ \dot{w} = -f(y,z,w)w\text{-}h(z)\text{-}g(y)\text{-}\alpha_4 x + p(t). \qquad (2)$$

Our main result is expressed in the following theorem:

Theorem

Assume that the following conditions hold:

(i) $f(y,z,w) \geqslant \alpha_1 > 0$ for all y,z,w and $\alpha_4 > 0$;
(ii) $h(0) = 0$ and $h(z)/z \geqslant \alpha_2 > 0$ $(z \neq 0)$;
(iii) $g(0) = 0$ and $g(y)/y \geqslant \alpha_3 > 0$ $(y \neq 0)$;
(iv) there is a finite constant $\triangle_0 > 0$, such that

$$\{\alpha_1 \alpha_2 - g'(y)\}\alpha_3 - \alpha_1 \alpha_4 f(y,z,0) \geqslant \triangle_0 \quad \text{for all } y,z;$$

(v) $g'(y) - g(y)/y \leqslant \delta_1$ for all $y \neq 0$, where $\delta_1 < 2\alpha_4 \triangle_0/(\alpha_1 \alpha_3^2)$;

(vi) $\frac{1}{z} \int_0^z f(y,\zeta,0)d\zeta - f(y,z,0) \leqslant \delta_2$ for all $z \neq 0$, where $\delta_2 < 2\triangle_0/(\alpha_1^2 \alpha_3)$;

(vii) $0 \leqslant \frac{h(z)}{z} - \alpha_2 \leqslant \frac{\alpha_3^3}{\alpha_4^2}\varepsilon'$ for all $z \neq 0$, where ε'

$$0 < \varepsilon' < \varepsilon \leqslant \min\{\frac{\alpha_4}{\alpha_3}, \frac{\triangle_0}{2\alpha_1 \alpha_3 D_0}, \frac{\alpha_3}{4\alpha_4 D_0}$$
$$(\frac{2\alpha_4\triangle_0}{\alpha_1 \alpha_3^2} - \delta_1), \frac{\alpha_1}{4D_0}$$
$$(\frac{2\triangle_0}{\alpha_1^2 \alpha_3} - \delta_2)\}, \tag{3}$$

$$D_0 = \alpha_1 \alpha_2 + \alpha_2 \alpha_3 \alpha_4^{-1};$$

(viii) $yf_y(y,z,0) \leqslant 0$ and $zf_y(y,z,0) \leqslant 0$ for all y,z;

(ix) $zf_w(y,z,w) + d_2 y f_w(y,z,w) \geqslant 0$ for all y,z,w, where $d_2 = \varepsilon + \alpha_4 \alpha_3^{-1}$;

(x) $\int_0^t |p(\tau)| d\tau \leqslant A < \infty$ for all $t \geqslant 0$.

Then, given any finite x_0, y_0, z_0 and w_0, there is a finite constant $D = D(x_0, y_0, z_0, w_0)$ such that the unique solution $x(t)$ of (1), which is determined by the initial conditions

$$x(0) = x_0,\ \dot{x}(0) = y_0,\ \ddot{x}(0) = z_0,\ \dddot{x}(0) = w_0,$$

satisfies

$$|x(t)| \leq D,\ |\dot{x}(t)| \leq D,\ |\ddot{x}(t)| \leq D,\ |\dddot{x}(t)| \leq D \qquad \text{for all } t \geq 0.$$

Remarks

1) When $f(\dot{x}, \ddot{x}, \dddot{x}) \equiv \alpha_1$, $h(\ddot{x}) = \alpha_2\ddot{x}$ and $g(\dot{x}) = \alpha_3\dot{x}$, then the hypotheses (i) – (ix) reduce to

$$\alpha_1 > 0,\ \alpha_2 > 0,\ \alpha_3 > 0,\ \alpha_4 > 0,\ \{\alpha_1\alpha_2 - \alpha_3\}\,\alpha_3 - \alpha_1^2\,\alpha_4 > 0,$$

which is the Routh-Hurwitz criterion for the asymptotic stability in the large of trivial solution of the equation

$$x^{(4)} + \alpha_1\dddot{x} + \alpha_2\ddot{x} + \alpha_3\dot{x} + \alpha_4 x = 0.$$

2) If $f(\dot{x},\ddot{x},\dddot{x}) \equiv \alpha_1$, $h(\ddot{x}) = \alpha_2\ddot{x}$, and $g(\dot{x})$ is arbitrary, then the hypotheses (i) – (ix) reduce to that of [3]

3) When $f(\dot{x},\ddot{x},\dddot{x}) = f(\ddot{x})$, $h(\ddot{x}) = \alpha_2\ddot{x}$ and $g(\dot{x})$ is arbitrary, then the theorem reduces to [4].

4) When $f(\dot{x},\ddot{x},\dddot{x}) = f(\dot{x},\ddot{x})$, $h(\ddot{x}) = \alpha_2\ddot{x}$ and $g(\dot{x})$ is arbitrary, then the theorem reduces to [1].

5) To exemplify the existence of functions with the desired properties, i.e.,

$$f(y,z,w) \geq \alpha_1 > 0 \qquad \text{for all } y,z,w$$

$$\frac{1}{z}\int_0^z f(y,\zeta,0)\,d\zeta - f(y,z,0) \leq \delta_2 \qquad \text{for all } y,z (z \neq 0)$$

f_y and f_w exists and are continuous for all y,z,w

$$yf_y(y,z,0) \leq 0 \ \text{ and } \ zf_y(y,z,0) \leq 0 \qquad \text{for all } y,z$$

$$zf_w(y,z,w) + d_2yf_w(y,z,w) \geq 0 \qquad \text{for all } y,z,w,$$

we give the following:

Let $f_1(z)$ be an arbitrary function with

$$\alpha_1 + 1 \leqslant f_1(z) < \alpha_2\alpha_3\alpha_4^{-1} \quad \text{for all z and}$$

$$\frac{1}{z}\int_0^z f_1(\zeta)d\zeta - f_1(z) \leqslant \delta_2 - \frac{1}{2} \quad \text{for all } z \neq 0$$

then the following function satisfies the above properties in the considered domains

$$f(y,z,w) = \begin{cases} f_1(z) - z(1 - e^{-y^2-w^2}) & \text{for } y \geqslant 0,\ 1 \geqslant z \geqslant 0,\ w \leqslant 0 \\ f_1(z) - (1 - e^{-y^2-w^2}) & \text{for } y \geqslant 0,\ z \geqslant 1,\ w \leqslant 0 \\ f_1(z) + z(1 - e^{-y^2-w^2}) & \text{for } y \leqslant 0,\ -1 \leqslant z \leqslant 0,\ w \geqslant 0 \\ f_1(z) - (1 - e^{-y^2-w^2}) & \text{for } y \leqslant 0,\ z \leqslant -1,\ w \geqslant 0. \end{cases}$$

Preliminary

Our treatment of the theorem is indirect, i.e., in terms of solutions of (2) above: Obviously, solutions of (2) above exist for $t \geqq 0$ and are uniquely defined by their initial conditions. So, let $(x(t), y(t), z(t), w(t))$ be the solution of (2) satisfying

$$x(0) = x_0,\ y(0) = y_0,\ z(0) = z_0,\ w(0) = w_0. \tag{4}$$

Then, the theorem will follow as soon as it will be shown that there is a constant $D = D(x_0, y_0, z_0, w_0)$ such that

$$|x(t)| \leqslant D,\ |y(t)| \leqslant D,\ |z(t)| \leqslant D,\ |w(t)| \leqslant D \quad \text{for all } t \geqslant 0. \tag{5}$$

An Auxiliary Function

Our proof of (5) depends entirely on the properties of an auxiliary function $V = V(x,y,z,w)$ defined by

$$\begin{aligned} 2V = {} & \alpha_4 d_2 x^2 + (\alpha_2 d_2 - \alpha_4 d_1) y^2 + 2\int_0^y g(\eta)\, d\eta \\ & + 2\int_0^z (d_1 h(\zeta) - d_2 \zeta)\, d\zeta + 2\int_0^z \zeta f(y, \zeta, 0)\, d\zeta \\ & + d_1 w^2 + 2\alpha_4 xy + 2\alpha_4 d_1 xz + 2d_2 y \int_0^z f(y, \zeta, 0)\, d\zeta \\ & + 2d_1 z g(y) + 2d_2 yw + 2zw \end{aligned} \tag{6}$$

where

$$d_1 = \varepsilon + \alpha_1^{-1},\ d_2 = \varepsilon + \alpha_4\alpha_3^{-1}, \tag{7}$$

$\varepsilon > 0$ being the constant in (3).

Now, let

$$\phi(y,z,0) = \begin{cases} \frac{1}{z}\int_0^z f(y,\zeta,0)d\,\zeta & \text{for } z \neq 0 \\ f(y,0,0) & \text{for } z = 0 \end{cases} \tag{8}$$

it follows from (i) and (vi) that

$$\phi(y,z,0) \geqslant \alpha_1 > 0 \quad \text{for all } y,z \tag{9}$$

$$\phi(y,z,0) - f(y,z,0) \leqslant \delta_2 \quad \text{for all } y,z. \tag{10}$$

Further, we define

$$\gamma(y) = \begin{cases} \frac{g(y)}{y} & \text{for } y \neq 0 \\ g'(0) & \text{for } y = 0 \end{cases} \tag{11}$$

Because of the assumptions (iii) – (v), we obtain

$$\gamma(y) \geqslant \alpha_3 > 0 \quad \text{for all } y \tag{12}$$

$$\{\alpha_1\alpha_2 - \gamma(y)\}\alpha_3 - \alpha_1\alpha_4 f(y,z,0) \geqslant \Delta_0 \quad \text{for all } y,z \tag{13}$$

$$g'(y) - \gamma(y) \leqslant \delta_1 \quad \text{for all } y. \tag{14}$$

Since the condition $g(0) = 0$ implies that $\gamma(y) = g'(\theta y)$ $(0 \leqslant \theta \leqslant 1)$, the inequality (13) is obvious from (iv).

The following two lemmas are essential for the proof of the theorem.

Lemma 1

Under the hypotheses (i) – (ix) of the theorem, $V(0,0,0,0) = 0$, and furthermore, there is a positive constant D_1 depending only on ε, α_1, α_2, α_3, α_4, δ_1, δ_2 and Δ_0 such that

$$V \geqslant D_1 (x^2 + y^2 + z^2 + w^2) \quad \text{for all } x,y,z,w. \tag{15}$$

Proof

We consider now the function V (x,y,z,w) defined by (6). Since g(0) = 0 it is obvious that V (0,0,0,0) = 0.

By (ii), we thus have that

$$\begin{aligned}
2V \geqslant{}& \alpha_4 d_2 x^2 + (\alpha_2 d_2 - \alpha_4 d_1) y^2 + 2 \int_0^y g(\eta)\, d\eta \\
&+ (\alpha_2 d_1 - d_2) z^2 + 2 \int_0^z \zeta f(y,\zeta,0) d\zeta \\
&+ d_1 w^2 + 2\alpha_4 xy + 2\alpha_4 d_1 xz + 2d_2 y \int_0^z f(y,\zeta,0) d\zeta \\
&+ 2d_1 zg(y) + 2d_2 yw + 2zw \\
={}& \{\alpha_4 d_2 - \frac{\alpha_4^2}{\gamma(y)}\} x^2 \\
&+ \{ \alpha_2 d_2 - \alpha_4 d_1 - d_2^2 \phi (y,z,0) \} y^2 + \{ 2 \int_0^y g(\eta) d\eta - y^2 \gamma(y) \} \\
&+ \{ \alpha_2 d_1 - d_2 - d_1^2 \gamma (y) \} z^2 + \{ 2 \int_0^z \zeta f(y,\zeta,0) d\zeta - z^2 \phi (y,z,0) \} \\
&+ \{ d_1 - \frac{1}{\phi (y,z,0)} \} w^2 + \frac{1}{\phi(y,z,0)} \{w + z\, \phi(y,z,0) + d_2 y \phi(y,z,0) \}^2 \\
&+ \frac{1}{\gamma(y)} \{\alpha_4 x + y\gamma (y) + d_1 z\gamma (y) \}^2.
\end{aligned}$$

From the definition of d_2 in (7) and from (12) it follows that

$$\{ \alpha_4 d_2 - \frac{\alpha_4^2}{\gamma (y)} \} x^2 \geqslant \alpha_4 \varepsilon x^2.$$

From the definition of d_1 in (7) and from (9), we obtain

$$\{ d_1 - \frac{1}{\phi (y,z,0)} \} w^2 \geqslant \varepsilon w^2$$

and hence,

$$2V \geqslant \alpha_4 \varepsilon x^2 + \varepsilon w^2 + V_1 + V_2 \tag{16}$$

where

$$V_1 = \{ \alpha_2 d_2 - \alpha_4 d_1 - d_2^2 \phi(y,z,0) \} y^2 + \{ 2 \int_0^y g(\eta)\, d\eta - y^2\gamma(y)\},$$

$$V_2 = \{ \alpha_2 d_1 - d_2 - d_1^2 \gamma(y) \} z^2 + \{ 2 \int_0^z \zeta f(y,\zeta,0) d\zeta - z^2 \phi(y,z,0) \}.$$

Using the relation

$$\int_0^y \eta g'(\eta)\, d\eta = yg(y) - \int_0^y g(\eta)\, d\eta$$

and the inequality (14) we get

$$\begin{aligned} 2 \int_0^y g(\eta)\, d\eta - y^2\gamma(y) &= \int_0^y g(\eta)\, d\eta - \int_0^y \eta g'(\eta)\, d\eta \\ &= \int_0^y \{ \gamma(\eta) - g'(\eta) \}\, \eta\, d\eta \\ &\geqslant -\frac{\delta_1}{2} y^2. \end{aligned}$$

Furthermore

$$\begin{aligned} \alpha_2 d_2 - \alpha_4 d_1 - d_2^2 \phi(y,z,0) &= d_2 \{\alpha_2 - d_1\gamma(y) - d_2\phi(y,z,0)\} + d_1 \{d_2 \gamma(y) - \alpha_4\} \\ &> d_2 \{\alpha_2 - d_1\gamma(y) - d_2\phi(y,z,0)\} \quad \text{by (7), (12)} \end{aligned}$$

and from the definition of ϕ in (8), we obtain

$$\phi(y,z,0) = f(y,\tilde{z},0), \text{ where } \tilde{z} = \theta z,\ 0 \leqslant \theta \leqslant 1.$$

Hence, by (7) and (13)

$$\begin{aligned} \alpha_2 - d_1\gamma(y) - d_2 f(y,\tilde{z},0) &= \{\alpha_2 - \frac{1}{\alpha_1}\gamma(y) - \frac{\alpha_4}{\alpha_3} f(y,\tilde{z},0)\} - \varepsilon\{\gamma(y) + f(y,\tilde{z},0)\} \\ &\geqslant \frac{\Delta_0}{\alpha_1\alpha_3} - \varepsilon\, (\alpha_1\alpha_2 + \alpha_2\alpha_3\alpha_4^{-1}) \\ &\geqslant \frac{\Delta_0}{\alpha_1\alpha_3} - \varepsilon\, D_0 \end{aligned} \tag{17}$$

and we obtain

$$V_1 \geqslant \frac{\alpha_4}{\alpha_3}\left(\frac{\Delta_0}{\alpha_1\alpha_3} - \varepsilon D_0\right) y^2 - \frac{\delta_1}{2}\, y^2$$

$$= \frac{1}{2}\left\{\frac{2\Delta_0\,\alpha_4}{\alpha_1\,\alpha_3^2} - \frac{2\alpha_4 D_0}{\alpha_3}\,\varepsilon - \delta_1\right\} y^2.$$

But by (v), $0 < \frac{2\alpha_4\Delta_0}{\alpha_1\,\alpha_3^2} - \delta_1$ and so,

$$V_1 \geqslant \frac{1}{4}\left\{\frac{2\alpha_0\Delta_0}{\alpha_1\,\alpha_3^2} - \delta_1\right\} y^2 \geqslant 0 \quad \text{by (3).}$$

In the same way from the relation

$$\int_0^z \zeta\, f(y,\zeta,0)\, d\zeta = z\int_0^z f(y,\zeta,0)\, d\zeta - \int_0^z \zeta\,\phi(y,\zeta,0)\, d\zeta$$

we get

$$\begin{aligned} 2\int_0^z \zeta f(y,\zeta,0)\, d\zeta - z^2\phi(y,z,0) &= \int_0^z \zeta f(y,\zeta,0) d\zeta - \int_0^z \zeta\phi(y,\zeta,0) d\zeta \\ &= \int_0^z \{f(y,\zeta,0) - \phi(y,\zeta,0)\}\,\zeta d\zeta \\ &\geqslant -\frac{\delta_2}{2}\, z^2 \quad \text{by (10).} \end{aligned}$$

Furthermore, by (7), (i) and (17)

$$\begin{aligned} \alpha_2 d_1 - d_2 - d_1^2\gamma(y) &= d_1\{\alpha_2 - d_1\gamma(y) - d_2 f(y,z,0)\} + d_2\{d_1 f(y,z,0) - 1\} \\ &\geqslant d_1\{\alpha_2 - d_1\gamma(y) - d_2 f(y,z,0)\} \\ &\geqslant d_1\left\{\frac{\Delta_0}{\alpha_1\alpha_3} - \varepsilon D_0\right\} \cdot \end{aligned}$$

We thus have

$$V_2 \geqslant \frac{1}{\alpha_1}\left\{\frac{\Delta_0}{\alpha_1\,\alpha_3} - \varepsilon D_0\right\} z^2 - \frac{\delta_2}{2} z^2$$

$$= \frac{1}{2}\left\{\frac{2\Delta_0}{\alpha_1^2\,\alpha_3} - \frac{2D_0}{\alpha_1}\,\varepsilon - \delta_2\right\} z^2$$

since by (vi), $0 < \frac{2\triangle_0}{\alpha_1^2\alpha_3} - \delta_2$ then (3) implies that

$$V_2 \geqslant \frac{1}{4}\left\{\frac{2\triangle_0}{\alpha_1^2\alpha_3} - \delta_2\right\} z^2 \geqslant 0.$$

Substituting these estimates of V_1 and V_2 in (16), we obtain

$$2V \geqslant \alpha_4\,\varepsilon\,x^2 + \frac{1}{4}\left\{\frac{2\,\alpha_4\triangle_0}{\alpha_1\,\alpha_3^2} - \delta_1\right\} y^2 + \frac{1}{4}\left\{\frac{2\,\triangle_0}{\alpha_1^2\,\alpha_3} - \delta_2\right\} z^2 + \varepsilon w^2$$

which proves that there is a positive constant D_1 such that

$$V \geqq D_1\,(x^2 + y^2 + z^2 + w^2).$$

Lemma 2

Under the hypotheses (i) - (ix) of the theorem, there exist positive constants D_2 and D_3 depending only on ε', ε, α_1, α_2, α_3, α_4 and $\triangle_0$, such that, if (x(t), y(t), z(t), w(t)) is any solution of (2), then

$$V \equiv \frac{d}{dt} V(x(t), y(t), z(t), w(t)) \leqq -D_2(y^2 + z^2 + w^2) + D_3(3 + y^2 + z^2 + w^2)\,|\,p(t)|.$$

Proof

Given any solution (x(t), y(t), z(t), w(t)) of (2), a straightforward calculation yields

$$\begin{aligned}\dot{V} &= \alpha_4 d_2 xy + (\alpha_2 d_2 - \alpha_4 d_1)yz + zg(y) \\ &\quad + w(d_1 h(z) - d_2 z) + z \int_0^z \zeta f_y(y, \zeta, 0)\,d\zeta + zwf(y,z,0) \\ &\quad + d_1 w\,\{ -f(y,z,w)w - h(z) - g(y) - \alpha_4 x + p(t)\} \\ &\quad + \alpha_4 y^2 + \alpha_4 xz + \alpha_4 d_1 yz + \alpha_4 d_1 xw \\ &\quad + d_2 z \int_0^z f(y, \zeta, 0)\,d\zeta + d_2 y\,\left\{ \int_0^z z f_y(y, \zeta, 0)\,d\zeta + wf(y,z,0)\right\} \\ &\quad + d_1 w g(y) + d_1 z^2 g'(y) + d_2 zw\end{aligned}$$

$$
\begin{aligned}
&+ d_2y\,\{-f(y,z,w)w - h(z) - g(y) - \alpha_4 x + p(t)\} + w^2 \\
&+ z\,\{-f(y,z,w)w - h(z) - g(y) - \alpha_4 x + p(t)\} \\
&= z \int_0^z \zeta f_y(y,\zeta,0)d\zeta - d_1 f(y,z,w)w^2 + \alpha_4 y^2 + d_2 z \int_0^z y f_y(y,\zeta,0)d\zeta \\
&+ d_2 z^2 f(y,\tilde{z},0) + d_1 g'(y)z^2 - d_2 y g(y) + w^2 - \alpha_2 z^2 \\
&- U - W + (d_1 w + d_2 y + z)p(t),\ \tilde{z} = \theta z\ (0 \leqslant \theta \leqslant 1).
\end{aligned}
$$

where

$$U = -\alpha_2 d_2 yz + d_2 y h(z) + z h(z) - \alpha_2 z^2,$$

$$W = \{f(y,z,w) - f(y,z,0)\}\,zw + \{f(y,z,w) - f(y,z,0)\}\,d_2 yw.$$

Because of (viii), it follows that

$$z \int_0^z \zeta f_y(y,\zeta,0)d\zeta \leqslant 0 \text{ and } d_2 z \int_0^z y f_y(y,\zeta,0)d\zeta \leqslant 0$$

and by (iii) $yg(y) \geqslant \alpha_3 y^2$.

Then, using these results we get

$$
\begin{aligned}
\dot{V}(t) \leqslant &-\{d_2\alpha_3 - \alpha_4\}y^2 - \{\alpha_2 - d_1 g'(y) - d_2 f(y,\tilde{z},0)\}z^2 \\
&- \{d_1\alpha_1 - 1\}w^2 - U - W + (d_1 w + d_2 y + z)p(t).
\end{aligned}
$$

From (7) it follows that

$$d_2\alpha_3 - \alpha_4 = \varepsilon\alpha_3\,,\ d_1\alpha_1 - 1 = \varepsilon\alpha_1$$

By (7) and (iv)

$$
\begin{aligned}
\alpha_2 - d_1 g'(y) - d_2 f(y,\tilde{z},0) &= \alpha_2 - \frac{1}{\alpha_1} g'(y) - \frac{\alpha_4}{\alpha_3} f(y,\tilde{z},0) - [g'(y) + f(y,\tilde{z},0)]\,\varepsilon \\
&\geqslant \frac{\Delta_0}{\alpha_1\alpha_3} - \varepsilon D_0 \geqslant \frac{\Delta_0}{2\alpha_1\alpha_3} \text{ by (3).}
\end{aligned}
$$

Then, we have

$$\dot{V}(t) \leqslant -\varepsilon\alpha_3 y^2 - \frac{\Delta_0}{2\alpha_1\alpha_3}z^2 - \varepsilon\alpha_1 w^2 - U - W + |d_1 w + d_2 y + z|\,|p(t)|. \tag{18}$$

Further, U = 0 when z = 0, and from (ii) if $z \neq 0$

$$\begin{aligned} U = & \left(\frac{h(z)}{z} - \alpha_2\right)(z^2 + d_2yz) \\ \geqslant & -\left(\frac{h(z)}{z} - \alpha_2\right)\frac{d_2^2y^2}{4}. \end{aligned}$$

Therefore, U satisfies

$$U \geqslant -\left(\frac{h(z)}{z} - \alpha_2\right)\frac{d_2^2y^2}{4} \quad \text{for all } y,z$$

and so, by (7) and (vii) the last result can be expressed as follows:

$$U \geqslant -\varepsilon' \alpha_3 y^2.$$

W = 0 when w = 0, and from (ix) if $w \neq 0$

$$\begin{aligned} W &= \left\{\frac{f(y,z,w) - f(y,z,0)}{w}\right\} zw^2 + \left\{\frac{f(y,z,w) - f(y,z,0)}{w}\right\} d_2yw^2 \\ &= \{zf_w(y,z,\mu w) + d_2yf_w(y,z,\mu w)\}\, w^2 \quad \text{where } 0 < \mu < 1 \\ &\geqslant 0. \end{aligned}$$

But

$$|d_2y + d_1w + z| \leqslant D_3(|y| + |z| + |w|)$$

where $D_3 = \max\{1, d_1, d_2\}$. Hence, since

$$|y| < 1 + y^2,\ |z| < 1 + z^2,\ |w| < 1 + w^2$$

for all y,z and w, we get

$$|d_2y + d_1w + z| \leqslant D_3(3 + y^2 + z^2 + w^2).$$

Then substituting these estimates of U,W and $|d_2y + d_1w + z|$ into (18) we obtain

$$\begin{aligned} \dot{V}(t) &\leqslant -(\varepsilon - \varepsilon')\alpha_3y^2 - \frac{\Delta_0}{2\alpha_1\alpha_3}z^2 - \varepsilon\alpha_1w^2 + D_3(3 + y^2 + z^2 + w^2)|p(t)| \\ &\leqslant -D_2(y^2 + z^2 + w^2) + D_3(3 + y^2 + z^2 + w^2)|p(t)| \end{aligned}$$

where $D_2 = \min\left\{(\varepsilon - \varepsilon')\alpha_3, \frac{\Delta_0}{2\alpha_1\alpha_3}, \varepsilon\alpha_1\right\}$. This completes the proof of the lemma.

Proof of the Theorem

The proof is based on a method devised by Antosiewicz [2]. We consider the function V given by (6). By Lemma 1, V satisfies

$$V(x,y,z,w) = 0, \quad x^2 + y^2 + z^2 + w^2 = 0$$

$$V(x,y,z,w) > 0, \quad x^2 + y^2 + z^2 + w^2 \neq 0$$

$$V(x,y,z,w) \to \infty \text{ as } x^2 + y^2 + z^2 + w^2 \to \infty.$$

Let $(x(t), y(t), z(t), w(t))$ be the solution of (2) satisfying the initial conditions (4) and consider the function $V(t) \equiv V(x(t), y(t), z(t), w(t))$ corresponding to this solution. From (15) it follows that to prove (5) it will be sufficient to show that there is a constant $D_6 = D_6(x_0, y_0, z_0, w_0) > 0$ satisfying

$$V(t) \leqslant D_6, \ t \geqslant 0.$$

By Lemma 2, it follows that

$$\dot{V}(t) \leqslant D_3 (3 + y^2 + z^2 + w^2) \mid p(t) \mid.$$

But from Lemma 1 again

$$y^2 + z^2 + w^2 \leqslant \frac{1}{D_1} V(x,y,z,w).$$

The we get that

$$\dot{V} - D_5 \mid p(t) \mid V \leqslant D_4 \mid p(t) \mid.$$

where $D_4 = 3D_3$ and $D_5 = D_3/D_1$.

Therefore we obtain the result

$$V(t) \leqslant \frac{1}{\chi(t)} \left(V(0) + D_4 \int_0^t |p(\tau)| \chi(\tau) \, d\tau\right),$$

where

$$\chi(t) \equiv \exp\left(- D_5 \int_0^t \mid p(\tau) \mid d\tau\right).$$

Since $\chi(t) \leqslant 1$ for $t \geqslant 0$, and by hypothesis (x)

$$V(t) \leqslant (V(0) + D_4 A) e^{D_5 A}, \ t \geqslant o.$$

Because $V(0) = V(x_0, y_0, z_0, w_0)$, we have for each $t \geqslant 0$

$$V(t) \leqslant D_6 ,$$

and hence the existence of $D = D(x_0, y_0, z_0, w_0) > 0$ is shown. The proof of the theorem is thus completed.

Conclusions

Let us note that the function V defined by (6) is a slight modification of the Lyapunov function considered by Ezeilo [4].

Although in the proofs of Lemma 1 and 2 we have mainly followed the same steps as in the article by Abou-El-Ela [1] the proof of our theorem is different. Since in obtaining the inequality (3.1) of [1] the derivative of the function $\sqrt{v(t)}$ may not exist for certain values of t.

Comparing our equation (1) with the equation

$$x^{(4)} + f_1(\ddot{x})\dddot{x} + f_2(\dot{x},\ddot{x}) + g(\dot{x}) + h(x,\dot{x}) = p(t)$$

which is considered by Lalli and Skrapek we note that there are certain differences between hypotheses concerning our Theorem and Theorem 2 in [5], in particular (v) in [5] and our hypotheses (vi) and (vii) are different. This is due to the fact that the functions V and the paths followed in their formations are not identical.

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حلول محدودة لمعادلة تفاضلية من المرتبة الرابعة

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ملخص البحث. تتضمن هذه الدراسة إيجاد الحلول لمعادلة غير خطية من الدرجة الرابعة وعلى الشروط الكافية التي تجعل هذه الحلول محدودة. والمعادلة التي أخذت أساسًا لهذه الدراسة هي:

$$x^{(4)} + f(\dot{x}, \ddot{x}, \dddot{x})\dddot{x} + h(\ddot{x}) + g(\dot{x}) + \alpha_4 x = P(t) \qquad (4)$$

لقد أعطيت نظرية للشروط الكافية التي تضمن وجود حلول محدودة لهذه المعادلة. تمهيداً لاثبات هذه النظرية فقد تم أولاً تعريف الاقتران $v(x, y, z, w)$. وبعد الوقوف على فرضيتان (Two Lemmas) تتعلقان بخواص هذا الاقتران، فقد تم إتمام الاثبات بمساعدتها، وقد أعطي إقتران عام كمثال يناسب النظرية ويناسب الافتراضات للاقتران $f(\dot{x}, \ddot{x}, \dddot{x})$ الذي تحويه المعادلة.