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Studies on the Method of Orthogonal Collocation III. The Use of Jacobi Orthogonal Polynomials for the Solution of Boundary Value Problems

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Abstract. Previous work identified the kind of Jacobi polynomials suitable to solve boundary value problems of the diffusion type in a slab, cylinder or sphere. In this paper we identify the general form of boundary value problems for which Jacobi orthogonal polynomials are optimal when they are used within the framework of the method of orthogonal collocation.

Introduction

For boundary value problems expressing diffusion with first order reaction, Villadsen and Michelsen [1] compared different methods of weighted residuals for their solution and concluded that Galerkin method is the best. A collocation method where the residual is required to be zero at the zeros of the proper Jacobi polynomial could give the same result as the Galerkin method. Villadsen and Michelsen [1] identified the proper Jacobi polynomial for a slab, cylindrical and spherical geometry. The question arises of what the general form of boundary value problems, for which a general Jacobi polynomial is the optimal choice. We address this question in this paper and give some general forms. Numerical results confirm the efficiency of the method of collocation as applied to this general form. In the previous two parts of this series [2,3] we applied the method of orthogonal collocation to the transient heat conduction problem.

Jacobi Orthogonal Polynomials

Jacobi polynomials $P_n^{\alpha\beta}(x)$ satisfy the orthogonality condition

$$\int_0^1 (1-x)^{\alpha} x^{\beta} P_n^{\alpha,\beta}(x) P_m^{\alpha,\beta}(x)\,dx = 0. \qquad m \neq n$$
$$\neq 0 \quad m = n, \qquad m = 1,2,\ldots n \qquad \alpha,\beta > -1 \qquad x \in [0,1] \ldots \tag{1}$$

For brevity we will write

$$P_n(x) = P_n^{\alpha,\beta}(x) \tag{2}$$

They satisfy the following differential Eq. [4].

$$x(1-x)\frac{d^2 P_n}{dx^2} + [(1+\beta)-(\alpha+\beta+2)x]\frac{dP_n}{dx} = -n(n+\alpha+\beta+1)P_n(x) \tag{3}$$

which can be put in the following form

$$\frac{1}{x^{\beta}}\frac{d}{dx}\left[x^{\beta+1}(1-x)^{1-\alpha}\frac{d}{dx}\{(1-x)^{\alpha} P_n(x)\}\right] = [-n(n+\alpha+\beta+1) - \alpha(1+\beta)]\, P_n(x) \tag{4}$$

Differential Equations for which Jacobi Polynomials are Optimal

Villadsen and Michelsen [1] concluded that Jacobi polynomials are optimal for problems of diffusion and chemical reaction with $\alpha = 1$, $\beta = -1/2, 0, 1/2$ for a slab, cylinder and sphere, respectively. The question arises if there are differential equations for which other values of α and β are optimal.

Let us have a look at the general differential equation

$$\frac{1}{x^{\beta}}\frac{d}{dx}\left[x^{\beta+1}(1-x)^{1-\alpha}\frac{d}{dx}\{(1-x)^{\alpha-1}(u-1)\}\right] + c\frac{(u-1)}{1-x} = \phi^2 u$$

or

$$x\frac{d^2 u}{dx^2} + \frac{[(\beta+1)(1-x)-(\alpha-1)x]}{1-x}\frac{du}{dx}$$

$$\frac{[(c-(\alpha-1)(\beta+1))(1-x)-(\alpha-1)x]}{(1-x)^2}(u-1)=\phi^2 u \tag{5}$$

and

$$u(1) = 1 \tag{6}$$

where c is a constant.
If we write the solution u(x) as

$$u(x) = 1 + (1-x)\sum_{i=0}^{N} a_i P_i(x) \tag{7}$$

where

$$P_i(x)=P_i^{\alpha,\beta}(x)$$

and a's are constants then substitute Eq. (7) into Eq. (5), we obtain

$$\frac{1}{x^{\beta}}\frac{d}{dx}\left[x^{\beta+1}(1-x)^{(1-\alpha)}\frac{d}{dx}(1-x)^{\alpha}\sum_{i=0}^{N}a_iP_i(x)\right]+c\sum_{i=0}^{N}a_iP_i(x)$$
$$=\phi^2\left[1+(1-x)\sum_{i=0}^{N}a_iP_i(x)\right] \tag{8}$$

By virtue of Eq. (4), the left hand side of Eq. (8) can be written as

$$\frac{1}{x^{\beta}}\frac{d}{dx}\left[x^{\beta+1}(1-x)^{(1-\alpha)}\frac{d}{dx}(1-x)^{\alpha}\sum_{i=0}^{N}a_iP_i(x)\right]+c\sum_{i=0}^{N}a_iP_i(x)$$
$$=\sum_{i=0}^{N}b_iP_i(x) \tag{9}$$

where the b's are constants related to the a's.

Thus the terms containing $P_N(x)$ in both the left hand side and the right hand side of Eq. (8) become zero if we use the zeros of $P_N(x)$ as collocation points. This is the desired property which makes this specific polynomial optimal since the solution will be accurate at the collocation points up to x^N whatever the value of a_N. Notice integral which can be evaluated accurately using Radau quadrature and which is similar to the effectiveness factor is

$$\eta = S\int_0^1 (1-x)^{\alpha-1} x^{\beta} u(x)\,dx \tag{10}$$

where

$$S = \frac{\Gamma(\alpha+\beta+1)}{[\Gamma(\alpha)\Gamma(\beta+1)]}$$

Since

$$\eta = S\int_0^1 (1-x)^{\alpha-1} x^{\beta}\, u(x)\,dx$$

$$= S\int_0^1 (1-x)^{\alpha-1} x^{\beta}\,[1+(1-x)\sum_{i=0}^{N} b_i P_i(x)]\,dx$$

$$= S\int_0^1 [(1-x)^{\alpha-1} x^{\beta} + (1-x)^{\alpha} x^{\beta} \sum_{i=0}^{N} b_i P_i(x)]\,dx$$

The term containing $P_N(x)$ will be zero if we use the zeros of $P_N(x)$ for the evaluation of the quadrature. Another general system of differential equations for which Jacobi polynomials are suitable is the following:

$$\frac{1}{x^{\beta}}\frac{d}{dx}\left[x^{\beta+1}(1-x)^{(1-\alpha)}\frac{d}{dx}[(1-x)^{\alpha}\frac{(u-(gx+f(1-x))}{(1-x)x}]\right]$$

$$+c\frac{[u(x)-(gx+f(1-x))]}{x(1-x)} = \phi^2 u$$

or

$$\frac{d^2u}{dx^2} + \frac{[(\beta-1)(1-x)-(\alpha-1)x]}{x(1-x)}\left[\frac{du}{dx} - g + f\right]$$

$$+\left[\frac{(1-\alpha)\beta}{x(1-x)} + \frac{(1-\beta)}{x^2} + \frac{(1-\alpha)}{(1-x)^2} + \frac{c}{x(1-x)}\right] \tag{11}$$

$$*[u-(gx+f(1-x))] = \phi^2 u$$

with

$$u(1) = g \tag{12}$$

and $$u(o) = f \tag{13}$$

If we write the solution u(x) as

$$u(x) = gx + f(1-x) + x(1-x) \sum_{i=0}^{N} bi\, Pi\,(x) \tag{14}$$

and make use of Eq. (3) we will note that both sides of Eq. (11) will have the term containing $P_N(x)$ zero if the collocation points are the zeros of $P_N(x)$. The following integral will be evaluated accurately using Lobatto quadrature,

$$\eta = S \int_0^1 x^{\beta-1} (1-x)^{\alpha-1}\, u(x)dx \tag{15}$$

where

$$S = \frac{\Gamma(\alpha+\beta)}{[\Gamma(\alpha)\,\Gamma(\beta)]}$$

Since

$$\eta = S \int_0^1 x^{\beta-1} (1-x)^{\alpha-1} [ax + d(1-x) + x(1-x) \sum_{i=0}^{N} b_i P_i\,(x)]\, dx$$

and the term containing $P_N(x)$ will be zero.

For problems of this kind, an arbitrary extra collocation point can be used to improve the profile. Also the dead zone method [1] can be used for large values of ϕ^2. In this case there will be a boundary layer near x = 1 and another one near x = 0. The dead zone will be in the middle.

A perturbation analysis of Eqs. (11-13) for small ϕ^2 will indicate that ϕ^2 term contains a cubic polynomial in x and ϕ^4 term contains a polynomial of fifth order in x. This is in contrary to the system (5-6) where the order of the polynomial increases by one as the order of ϕ^2 increases by 2, (see appendix A)

Thus we need more collocation points for the system (11-13) to get the same accuracy in ϕ^2 as for system (5 and 6). Thus it is desirable to identify cases for which Eqs. (11-13) can be transformed into equations (5 and 6). The following two cases are identified.

The following transformation

$$u(x) = x\,v(x)$$
$$y = x^2$$

will transform Eqs. (11-13) into

$$4y\frac{d^2v}{dy^2} + 2(\beta+2)\frac{dv}{dy} = \phi^2 v$$

$$v(1) = g$$

(ii) $g = f = 1$, and $\alpha = \beta$

In this case we use the transformation

$$y = (1 - 2x)^2$$

to obtain

$$y\frac{d^2u}{dy^2} + \left(\frac{(\beta-1)y}{1-y} + 0.5\right)\frac{du}{dy} + \frac{(1-\beta)(\beta+2)+c}{4(1-y)}$$
$$+\frac{(1-\beta)y}{(1-y)^2} = \frac{\phi^2}{16}u$$

Finally the system of differential equation

$$x(1-x)\frac{d^2u}{dx^2} + [1+\beta) - (\alpha+\beta+2)x]\frac{du}{dx} = -\phi^2\,u(x) \tag{16}$$

will have a solution other than the trivial solution $u(x) = 0$ if

$$\phi^2 = n\,(n+\alpha+\beta+1) \tag{17}$$

and

$$u(x) = b\,P_n(x) \tag{18}$$

Thus ϕ^2 will be the eigenvalues and $P_n(x)$ are the eigenvectors.
The following integral

$$\eta = \int_0^1 (1-x)^{\alpha} x^{\beta} u(x)dx \tag{19}$$

equal zero (if $n \neq 0$) and this will be true if the collocation points are taken as the zeros of $P_n(x)$ and we use Gauss quadrature.

Numerical Results

In this section we give numerical results for three examples. The results show that we can get high accuracy with low order polynomials. Examples (1) and (2) are linear boundary value problems. Although the development for this paper is for a linear system, example (3) shows that good results can also be obtained when we apply the collocation method to a non-linear problem.

Example 1

For Eq. (5) and the case of $c = 0$ and

i) $\alpha = 2, \beta = 4$

ii) $\alpha = 4, \beta = 2$

iii) $\alpha = 3, \beta = 3$

and $\phi^2 = 8$, obtain the numerical solution for u and η for $N = 1$ to 4.

In the following table, we list the values of u (o) and η

		N=1	N=2	N=3	N=4
$\alpha=2, \beta=4$	u(o)	0.3846	0.5256	0.5082	0.5081
	η	0.8242	0.8231	0.8231	0.8231
$\alpha=4, \beta=2$	u(o)	0.5294	0.5852	0.5805	0.5808
	η	0.7311	0.7300	0.7300	0.7300
$\alpha=3, \beta=3$	u(o)	0.5000	0.5814	0.5729	0.5735
	η	0.7857	0.7847	0.7847	0.7847

As expected η is calculated accurately with few collocation points.

Example 2

A first order reaction of a gas diffusing in a liquid film is given by

$$\frac{d^2u}{dx^2} = \phi^2 u$$

$$u(1) = g$$

$$u(0) = f$$

The solution is given by

$$u = \frac{g \sinh \phi x + f \sinh \phi (1-x)}{\sinh \phi}$$

compare the numerical solution of u,

$$D = \left.\frac{du}{dx}\right|_{x=1} \text{ and } I = \int_0^1 u\, dx$$

with the analytical solution for the case of $\phi^2 = 10$, $g = 1$, $f = 0.1$.

We first note that the above differential equation is obtained from the general differential Eq. (11) by letting $c = 0$, $\alpha = 1$ and $\beta = 1$. The following results are obtained:

x	u Analytical	U Calculated		
		N = 2	N = 4	N = 6
0.0000	0.10000	0.1000	0.1000	0.1000
0.1000	0.10000	0.1068	0.1001	0.1000
0.2000	0.1102	0.1140	0.1100	0.1102
0.3000	0.1314	0.1285	0.1313	0.1314
0.4000	0.1659	0.1570	0.1660	0.1659
0.5000	0.2171	0.2063	0.2174	0.2171
0.6000	0.2902	0.2830	0.2903	0.2903
0.7000	0.3926	0.3940	0.3924	0.3926
0.8000	0.5346	0.5460	0.5344	0.5346
0.9000	0.7305	0.7458	0.7306	0.7305
1.0000	1.0000	1.0000	1.0000	1.0000
I	0.3196	0.3208	0.3196	0.3196
D	3.1468	2.8375	3.1364	3.1466

Example 3

In this example we treat a non-linear problem where the R.H.S of Eq. (5) equals to $20\ u^2$.

For the case of c = 0 and (i) $\alpha = 2$, $\beta = 4$; (ii) $\alpha = 4$, $\beta = 2$; (iii) $\alpha = 3$, $\beta = 3$, we obtain the numerical solution for u (0) and η for N = 1 to N = 10.

In the following table, we list the values of u(0) and η .

		N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8	N=9	N=10
$\alpha = 2$	u(0)	0.1111	0.5967	0.3755	0.4626	0.4319	0.4419	0.4388	0.4397	0.4395	0.4395
$\beta = 4$	η	0.5767	0.5532	0.5517	0.5516	0.5516	0.5516	0.5516	0.5516	0.5516	0.5516
$\alpha = 4$	u(0)	0.3777	0.5375	0.4970	0.5068	0.5046	0.5051	0.5050	0.5050	0.5050	0.5050
$\beta = 2$	η	0.4271	0.4109	0.4102	0.4102	0.4102	0.4102	0.4102	0.4102	0.4102	0.4102
$\alpha = 3$	u(0)	0.2980	0.5631	0.4729	0.5007	0.4928	0.4949	0.4944	0.4945	0.4945	0.4945
$\beta = 3$	η	0.5039	0.4845	0.4835	0.4835	0.4835	0.4835	0.4835	0.4835	0.4835	0.4835

We notice that η is obtained accurately for N = 3. However a large number of collocation points are needed ($N \cong 9$) to obtain accurate value for u(0). Certainly the efficiency of the method will be affected by the degree of non-linearity as with any other numerical method.

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Appendix A

Substituting an expansion of the form

$$u = u_0 + \phi^2 u_1 + \phi^4 u_2 + \ldots \tag{A.1}$$

in Eqs. (5,6) and collecting terms of the same order in ϕ, we obtain

$$u = 1 - \frac{\phi^2 (1-x)}{((\beta+1)\alpha - c)} + \frac{\phi^4 (1-x)\left(\frac{(\beta+2)\alpha+1-c}{(\beta+1)\alpha-c} - x\right)}{((\beta+1)\alpha - c)((\beta+2)(\alpha+1)-c)} \tag{A.2}$$

As can be seen the term containing ϕ^2 is first order in x and the term containing ϕ^4 is second order in x.

The root of $P_1^{\alpha,\beta}(x_a)$, x_a is given by

$$X_a = \frac{\beta+1}{\beta+\alpha-2} \tag{A.3}$$

If we substitute x_a for x in Eq (A.2), we obtain,

$$u_a = 1 - \frac{\phi^2(\alpha+1)}{(\alpha+\beta+2)((\beta+1)\alpha-c)} + \frac{\phi^4(\alpha+1)^2}{(\alpha+\beta+2)^2((\beta+1)\alpha-c)^2}$$

where u_a is the value of u at x_a. We notice that the first three terms form a geometric sequence. If we assume that further terms will follow the same pattern, we can sum the geometric series to obtain

$$u_a = \frac{1}{1 + \dfrac{\phi^2(\alpha+1)}{(\alpha+\beta+2)((\beta+1)\alpha-c)}}$$

On the other hand if we substitute an expression of the form

$$u = 1 + \frac{(u_c - 1)(1-x)}{(1-x_a)} \tag{A.5}$$

in Eq. (5) and apply the collocation method at the root of $P_1^{\alpha,\beta}(x)$, x_a , we obtain

$$u_c = \frac{1}{1+\dfrac{\phi^2\,(\alpha+1)}{(\alpha+\beta+2)((\beta+1)\,\alpha-c)}} \tag{A.6}$$

We notice that u_c from (A.6) is the same as u_a in (A.4). Thus the one-point collocation is equivalent to summing up the regular perturbation expansion of u at the collocation point as if it is a geometric series and for a problem of the type of Eq. (5).

Now substitute the regular perturbation expansion (A.1) in Eqs. (11-13) and collecting terms of the same order in ϕ, we obtain

$$u = gx + f\,(1-x) - \phi^2 x(1-x)\Bigg[\Bigg(\frac{\dfrac{(g-f)(1-\beta)}{(1-\alpha)\,\beta+c-2\,(1+\alpha+\beta)} - f}{c-(1+\beta)\,\alpha} - \frac{(g-f)\,x}{(1-\alpha)\,\beta+c-2\,(1+\alpha+\beta)} + \ldots \tag{A.7}$$

We notice here that the zero order term is first order in x and the term containing ϕ^2 is third order in x. Similarly the term containing ϕ^4 will be fifth order in x, and so on.

دراسات على طريقة التنظيم المتعامد
٣ – استخدام تنظيم كثيرات الحدود المتعامد لجاكوبي لحل المسائل ذات القيمة الحدّية

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الرياض ١١٤٢١ ، المملكة العربية السعودية

(استلم في ١٠/١٢/١٩٩٧ م ، وقبل للنشر في ٢٢/٩/١٩٩٨ م)

ملخص البحث. تمّ، في أعمال سابقة ، تحديد نوع كثيرات الحدود لجاكوبي المناسبة لحل مسائل ذات قيم حدّية من نوع الانتشار في جسم على شكل لوح أو أسطوانة أو كُرة ، بينما يتم في هذه المقالة تحديد الشكل العام للمسائل ذات القيم الحدّية التي يكون حلها مناسباً باستخدام طريقة التنظيم المتعامد وكثيرات حدود جاكوبي .