Some Orbits of Nonsymmetric Isotropy Irreducible Coset Spaces

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Abstract. This paper deals with the calculation the volume function for orbits of nonsymmetric isotropy irreducible coset spaces. This paper also studies the volume function for orbits of regular elements of these spaces. The compact examples presented here are special cases. Also in this paper we find the critical points of this function, and when the orbits are minimals or maximals.

Introduction

A Lie-subgroup H of a Lie group G by adjoint action $\hat{h}\ g = hgh^{-1}$, $h \in H$ and $g \in G$. The orbit of the element $g \in G$ is denoted by O (g) and given by $O(g)=\{hgh^{-1}: h\in H\}$ and is a sub-manifold of the Riemannian manifold G equipped with the bi-invariant metic. The aim of the present paper is to find the volume of O(g) which we denote by Vol O (g).

Let M be Riemannian and G be a compact, connected Lie group of isometries of M ($G \subset I(M)$). The Riemannian metric g_{ij} on G is called bi-invariant if L_x, and R_x are isometries with respect to g_{ij} for all $x\in G$. If the scalar product (X,Y) on T_eG, is invariant with respect to the representation Ad_x, then it can be used for constructing a bi-invarant metric on the Lie group G. The metric $(\xi,\zeta)_x = (L_{x^{-1}}{}^*\xi, L_{x^{-1}}{}^*\zeta)$, for ζ, and $\zeta\in T_x(G)$ is a Riemannian metric on G (ct. [1-2]. The above metric is invariant with respect to all left translations, i.e., the metric (X, Y) is invariant with respect to the adjoint representation Ad_x, $x\in G$:$(X, Y)=(Ad_xX, Ad_x Y)$; X and $Y \in T_eG$, this implies that it is a bi-invariant metric on G. It is easy to show that the scalar product $(X, Y)_K$ is invariant with respect to the adjoint representation Ad_x, $x\in G$ on the Lie algebra T_eG of G, which is defined by: $(X, Y)_K= -\mathrm{tr}\ ad_x ad_y$; X and $Y\in T_eG$, and is called the killing's scalar product. This metric is invariant, i.e., $(Ad_xX, Ad_xY)_K=(X, Y)_K$; X and $Y\in T_eG$.

Theorem [3]. Let $g\in G$, be any element in the compact matrix Lie group G, H a closed subgroup, which acts on G by an adjoint action, and $\Gamma(g)=\{h\in H; hg=gh\}$ is a centralizer of g under this action. Let $e_1,\ldots, e_n$ be a bases of the Lie algebra of the subgroup H by mode algebra C (g) of Γ(g). Then

$$\text{Vol } O(g) = \frac{\text{Vol } H}{\text{Vol } (g)} \sqrt{\frac{\det\|(f_i, f_j)_k\|}{\det\|(e_i, e_j)_k\|}}$$ where $f_i = g^{-1}e_i g - e_i$, and $(XY)_K$ is a killing`s metric on the Lie algebra of G.

Volume Function of Irreducibily Homogenuous Space

Let W be a Riemannian space and G a compact, connected subgroup of isometries of W, i.e. $G=I_0(W)$. A Riemannian space W is called irreducible homogeneous, if W is a factor space W=G/H, and the Lie algebra of the Lie subgroup H acts irreducibility on the tangent space (ct. [4; 5].

Wolf [6] obtained a classification of such spaces. All irreducible spaces W prescribe a semi-simple compact Lie group H, which has a representation $\rho : H \to GL(W)$, and a simple compact Lie group G, where $\rho(H) \subset G \subset GL(W)$. Now if $\rho(H)$ acts on G in the following form: for every element $h \in \rho(H)$ there corresponds a ansformation $\hat{h}: G \to G$, where $\hat{h}(x)=hxh^{-1}$, $x \in G$. Let O)g) be the orbit for this action i.e., $O(g)=\{\hat{h}(g): h \in \rho(H)=\{hgh^{-1}; h \in \rho(H)\}=\{\rho(h)g(\rho(h))^{-1}; h \in$ the Lie algebra of H$\}$.

The general method for computation of the volume function of the irreducible homogeneous space as described in [3] is as follows:

1. Select a basis $e_i,\ldots, e_N$ in the representation space.
2. Select a basis $\tilde{h}_i,\ldots, \tilde{h}_g$ of the Lie algebra of the Lie group H.
3. Compute a basis $h_i,\ldots, h_g$ of the Lie algebra of the Lie group ρ(H).

All operators has a matrix expression: $d\rho(\tilde{h}_i)e_j = \alpha_y^k \; e_k$; $d\rho(\tilde{h}_y) = \|\alpha_y^k\|$, by the basis $e_i,\ldots,e_N$. Select the basis $h_{i1},\ldots, h_{ik}$, $k \le N$ from the basis $h_j,\ldots, h_S$ by module the centralizer (g) of a regular element $g \in G$. A regular element of the Lie group G belongs to a fixed maximum torus $T \subset G$.

4. Compute the matrices $g_l = g^{-1}h_{il}g - h_{il}$, $l = 1,2,\ldots,k$.
5. Compute det $(h_{il}, h_{jm}) = \beta$, and $\det(f_i, f_j) = \alpha$.
6. Compute the volume of the orbit i.e., Vol O (g) = Const $\sqrt{\frac{\alpha}{\beta}}$.

Let $h_j = U(m+1)$ acting on the space of tensors a_j^i; i,j = 1,2,....., m+1, with trace equal zero, where $a_j^i = -a_i^j$. This action preserved all quadratic foms $\sum a_j^i a_i^j$, and this implies that $\rho(h_j)$ is a subgroup in the group of transformation of the space V with determinent equal one. Here ρ:U(m+1)$\to$GL(V) which corresponds the representation of N=dim V = m^2+2m.

Theorem 1. Let g= diag ($e^{i\phi j}$,...., $e^{i\phi N}$) be a regular element of the group SO(N) (i.e., $\phi_i \neq \phi_j$; $i \neq j$). Then for m=1,

Vol O(g) = Const $\left\{(\cos(\phi_1-\phi_2)-1)^2+\sin^2(\phi_1-\phi_2)\right\}$; where O(g) is the orbit of the element g $\in$ SO(N) with respect to representation of the acting group ρ(U(m+1) on the group SO(N).

Proof: We follow the following steps for computation of the volume in this example.

a. We have the representation ρ:U(m+1)$\to$ GL(V), N= dim V, V space of representations (space of corresponding tensors). A base in the space V

$$\{e_g=\begin{bmatrix}0 & 1\\ 1_i & 0_j\end{bmatrix}^i j \;\; c_{ij}=\begin{bmatrix} & i\\ i_i & j\end{bmatrix}^i j \;\; e_{ij}=\begin{bmatrix}i & \\ i & i_i\end{bmatrix}^i j\;\} ,$$

where N=m^2+2m.

b. The general method for computation of the volume of orbit O(g).

We find the centerlizer $\Gamma(g)=\{h\in\rho(\hat{h}j)/gh=hg\}$ of the element g in the Lie group ρ(U(m+1)), and take the basis e_i, ..., e_N of the Lie algebra L (ρ(U(m+1)) by modulo Lie algebra L(Γ(g))=C(g). We have the equality C(g)=dρ(L(T)).

Therefore for construction corresponding to a basis e_i, ..., e_N in the representation space; must take a base of Lie algebra U(m+1) by mudulo Lie algebra L(T), and carry it with the help of dρ in L(ρ(U(m+1)).

c. A basis of Lie algebra U(m+1) by modulo Lie algebra of standard maximal torus;

which in the form: $\{\tilde{h}_{ij}^{(1)}=\begin{bmatrix} & +1\\ i_i & j\end{bmatrix}^i j\,,\;\tilde{h}_{ij}^{(2)}=\begin{bmatrix} & +i\\ i_i & j\end{bmatrix}^i j\}$

d. We calculate the base of Lie algebra d ρ(U(m+1):

$$d\;\rho(\ddot{h}_{ij}^{(1)})=h_{ij}^{(1)},\; d\;\rho(\tilde{h}_{ij}^{(2)})=h_{ij}^{(2)};\; d\;\rho(\tilde{h}_{ij}^{(k)})\;e_p=\sum_{l=1}^{N}\alpha_{ijp}^{ki}\;e_i\;;\;\alpha_{ijp}^{ki}\in C$$

e. We calculate the matrix

$f_{ij}^{(k)} = g_0^{-1}\;h_{ij}^{(k)}\;g_0 - h_{ij}^{(k)},\; g_0 = \text{diag}\;(c^{i\phi i},\ldots.,\;e^{i\phi N})$.

We computa $\det(h_{ij}^{(k)},\;h_{pq}^{(m)})=\beta$, $\det(f_{ij}^{(k)},\;f_{pq}^{(m)})=\alpha$

In this case . Vol O(g) = Const $\sqrt{\frac{\alpha}{\beta}}$.

To demonstrate the calculation we take m=1, then N=3, h_j=U(3) and G=SO(3). This means that the representation ρ must be defined by $\rho(h)X=hXh^{-1}$, from U(2) into GL (End R^2), where $h \in U(2)$; and $X \in$ End R^2=V. The derivative of the above representation can be written in the form: $d\rho(h)\,X=hX - Xh = [h,X]$.

Choose a basis in U(2): $\{\bar{h}_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix};\ \bar{h}_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}\}$, and the basis in

$$V = \text{End}\ R^2:\ \{X_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ X_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix},\ X_3 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\}.$$

Now, calculate $d\rho(\tilde{h}_1)$, and $d\rho(\bar{h}_2)$:

1. We compute the values:

$$d\rho\tilde{h}_1(X_1) = \tilde{h}_1X_1 - X_1\tilde{h}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$d\rho\tilde{h}_1(X_2) = \tilde{h}_1X_2 - X_2\tilde{h}_1 = \begin{bmatrix} 2i & 0 \\ 0 & 2i \end{bmatrix} = 2X_3$$

$$d\rho\tilde{h}_1(X_3) = \begin{bmatrix} 0 & 2i \\ -2_i & 0 \end{bmatrix} = -2X_2$$

2. We determine the element h_1:

$$d\rho\tilde{h}_1(X_1) = \lambda_1X_1 + \lambda_2X_2 + \lambda_3X_3,$$
$$d\rho\tilde{h}_1(X_2) = \mu_1X_1 + \mu_2X_2 + \mu_3X_3,$$
$$d\rho\tilde{h}_1(X_3) = \alpha_1X_1 + \alpha_2X_2 + \alpha_3X_3,$$

From this system we have $h_1 = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix}$

Similarly; $d\rho\bar{h}_2(X_1) = -2X_3$, $d\rho\tilde{h}_2(X_2) = 0$, and $d\rho\tilde{h}_2(X_3) = 2X_1$,

and obviously; $h_2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}$

This means that the basis of Lie algebra $d\rho(H)$ can be written in the form:

$$\{h_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix},\ h_2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}\}.$$

The value of $(h_1,\ h_2) = \text{tr}\ h_1h_2 = -8$, and similarly

$(hl_2,\ h_2) = -8$ and $(h_1, h_2) = (h_2,\ h_1) = 0$. Then

$$\left\|(h_i,\ h_j)\right\|_{i,j=1,2} = \begin{vmatrix} (h_1,\ h_1) & (h_1,\ h_2) \\ (h_2,\ h_1) & (h_2,\ h_1) \end{vmatrix}$$

Now, let g=diag $(e^{i\phi_1},\ e^{i\phi_2})$ be a regular element of U(2) then,

$$\rho(g)X_1 = gX_1g^{-1} = \begin{bmatrix} 0 & e^{i(\phi_1-\phi_2)} \\ -e^{-i(\phi_1-\phi_2)} & 0 \end{bmatrix} = \lambda X_1 + \mu X_2 + \alpha X_3.$$

To determine the values α, μ, λ : It is clear that $\alpha=0$ and from the solution of the system: $e^{i(\phi_1-\phi_2)} = \lambda + \mu i,\ -e^{i(\phi_1-\phi_2)} = -\lambda + \mu i$, we have

$$\mu = \frac{e^{i(\phi_1-\phi_2)} - e^{-i(\phi_1-\phi_2)}}{2i} = (\phi_1 - \phi_2);\ \text{and}\ \ \lambda = \sin\ (\phi_1 - \phi_2),\ \text{i.e.}$$

$$\rho(g)X_1 = \cos\ (\phi_1 - \phi_2)X_1 + \sin\ (\phi_1 - \phi_2)X_2 + 0X_3 \tag{1}$$

By the same way;

$$\rho(g)X_2 = \sin\ (\phi_1 - \phi_2)X_1 + \cos\ (\phi_1 - \phi_2)X_2 + 0X_3 \tag{2}$$

and

$$\rho(g)X_3 = 0X_1 + 0X_2 + X_3 \tag{3}$$

From (1), (2) and (3), we have:

$$\rho(g) = \begin{bmatrix} \cos\ (\phi_1-\phi_2) & -\sin\ (\phi_1-\phi_2) & 0 \\ \sin\ (\phi_1-\phi_2) & \cos\ (\phi_1-\phi_2) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Finally, we know that: $f_i = \rho(g)h_i\rho(g)^{-1} - h_i;\ i = 1, 2$, then

$$f_1 = \begin{bmatrix} 0 & 0 & -2\ \sin(\phi_1-\phi_2) \\ 0 & 0 & -2(\cos(\phi_1-\phi_2)-1 \\ -2\ \sin(\phi_1-\phi_2) & 2(\cos(\phi_1-\phi_2)-1 & 0 \end{bmatrix},\ \text{and}$$

$$f_2 = \begin{bmatrix} 0 & 0 & -2\ (\cos(\phi_1-\phi_2)-1) \\ 0 & 0 & -2\ \sin(\phi_1-\phi_2) \\ -2\ \cos(\phi_1-\phi_2)-1 & -2\ \sin(\phi_1-\phi_2) & 0 \end{bmatrix}$$

This implies that

$$(f_1,\ f_1) = \text{spur}\ \ f_1,\ f_1\ =\ 8\{\sin^2(\phi_1-\phi_2)+(\cos(\phi_1-\phi_2)-1)^2\};$$

$$(f_2,\ f_2) = -8\Big\{(\cos(\phi_1-\phi_2)-1)^2+\sin^2(\phi_1-\phi_2)\Big\},\ \text{and}$$

$$(f_1,\ f_2) = (f_2,\ f_1) = 0.$$

It is clear that

$$\left\|(f_i,\ f_j\right\|_{ij=1,2} = 16\Big\{(\cos(\phi_1-\phi_2)-1)^2+\sin^2(\phi_1-\phi_2)\Big\}^{2,}\ \ \text{and}$$

$$\text{Vol}\ \ 0(f) = \text{Const}\sqrt{\frac{\|(f_i,\ f_j)_k\|}{\|(h_i,\ h_j)_k\|}\quad i,j=1,2}$$

Table series of the volume function of orbits of the regular elements for some nonsymmetric isotropy irreducible spaces

h_i	V	G	G/h_1	g	Vol 0(g)
SU(4)	$a^{ij} = -a^{ij}$	SU(6)	$SU(6)/_{\rho}\ SU(4)$	$\text{diag}(e^{i\phi_1},\ldots e^{i\phi_6})$; $\phi_i \neq \phi_j;\ i \neq j.$	$\text{Const} \prod\limits_{\substack{1\le k<l\le 6\\ k+l<7}} (e^{i(\phi_k-\phi_l)}-1)$ $(e^{i(\phi_l-\phi_k)}-1)+(e^{i(\phi_{Ll}-\phi_{Lk})}-1)$ $(e^{i(\phi_{2k}-\phi_{2l})}-1)$
SU(4)	$a^{ij} = a^{ij}$ $i,\ j=1,\ldots,3.$	SU(6)	$SU(6)/_{\rho}SU(3)$	$\text{diag}(e^{i\phi_1},\ldots e^{i\phi_{10}})$; $\phi_i \neq_j;\ i\neq j.$	$\text{Const}(2A_{12}+2A_{24}+A_{34}).$ $(2A_{13}+2A_{36}+A_{25}).$ $(2A_{45}+2A_{56}+A_{23});$ $A_{ki} = (e^{i(\phi_k-\phi_l)}-1).$ $(e^{i(\phi_l-\phi_k)}-1);$ $l,\ k=1,\ldots,6.$
SU(4)	$a^{ij} = a^{ij}$ $i,j=1,\ldots,4.$	SU(10)	$SU(10)/_{\rho}SU(4)$	$\text{diag}\ (e^{i\phi_1},\ldots e^{i\phi_{10}})$; $\phi_i \neq \phi_j;\ i\neq j.$	$\text{Const}(2A_{15}+2A_{25}+A_{68}+A_{79})$ $(2A_{16}+2A_{36}+A_{58}+A_{7,10}).$ $(2A_{17}+2A_{47}+A_{59}+A_{6,10}).$ $(2A_{28}+2A_{38}+A_{56}+A_{9,10}).$ $(2A_{29}+2A_{49}+A_{57}+A_{8,10}).$ $(2A_{3,10}+2A_{4,10}+A_{67}+A_{89}).$ $A_{ki} = (e^{i(\phi_k-\phi_l)}-1).$ $(e^{i(\phi_l-\phi_k)}-1);$ $k,l=1,\ldots,10.$

Table series of the volume function of orbits of the regular elements for some nonsymmetric isotropy irreducible spaces

h_i	V	G	G/h_i	g	Vol 0(g)
SU(2) XSU(3)	$a^{i\alpha}$ $i=1,2;$ $\alpha=1,\dots,3.$	SU(6)	$SU(6)/\rho SU(2)$ xSU(3)	$diag(e^{i\phi_2},\dots,e^{i\phi_6});$ $\phi_i \neq \phi_j\ ;\ i\neq j.$	$Const(A_{12}+A_{45})(A_{23}+A_{56})$ $(A_{13}+A_{46})(A_{14}+A_{25}+A_{36});$ $A_{ki}=(e^{i(\phi_k-\phi_l)}-1).$ $(e^{i(\phi_l-\phi_k)}-1);$ $k,\ l=1,\dots,6.$
SU(1)	$a^i_j=a^j_i$ $i,j=1,2,$ with trac $=0.$	SO(3)	$SO(3)/\rho U(1)$	$diag\ (e^{i\phi_1},e^{i\phi_2},e^{i\phi_3});$ $\phi_i\neq\phi_j;\ i\neq j.$	$Const\ \{Cos(\phi_1-\phi_2)-1)^2+$ $(Sin^2(\phi_1-\phi_3)-1)\}$

To study the case, when this orbit is a minimal or a maximal orbit:

Let $f(\phi_1-\phi_2)=Const\{\phi_1-\phi_2)-1)^2+\sin^2(\phi_1-\phi_2)\}$ which is a maximal or minimal when (1-cos($\phi_1-\phi_2$) is a maximal or minimal. This means that when $(\phi_1-\phi_2)=\pi-2\pi k$, $k\in Z$ the set of integers.

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بعض مدارات فضاءات التجزئة المتشاكلة متريًّا وغير المتماثلة وغير القابلة للاختزال

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(سُلِّم في ١٩ جمادى الأولى ١٤١٤هـ؛ وقُبل للنشر في ٢٣ ذي الحجة ١٤١٥هـ)

ملخص البحث. تمَّ حساب مدارات العناصر المنتظمة للمانيفولد الريماني المتشاكل متريًّا وغير المتماثل وغير القابل للاختزال W والذي يمكن وضعه على الصورة W = G / H حيث G زمرة لي متراصة ومترابطة من التشاكل المتري للمانيفولد W وزمرتها الجزئية لي المغلقة H التي تحقق شرط كرتان. ولهذا الغرض استخدمت زمرات المصفوفات وزمراتها الجزئية التي تحقق الشروط السابقة. وتمَّ تعيين المدارات التي تحقق العلاقة Vol S′⊂S حسابيًّا. كما عيّنت النقط الحرجة التي تكون عندها هذه المدارات أكبر ما يمكن أو أصغر ما يمكن.