

# CIVIL ENGINEERING

## Non-Linear Finite Element Analysis for RC Elements Subjected to In-Plane Loads

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**Abstract.** A non-linear iterative solution of finite element formulation implemented a 486 personal computer for predicting the cracking shapes and their propagation in reinforced concrete elements subjected to in-plane stresses is presented. The model is based on a smeared crack representation and a plane stress state simulation. It considers the influence of concrete tension stiffening after cracking, compressive stress degradation due to cracking, and single and double crack conditions. The reliability of the model is checked against the experimental results for specimens that were tested by the author as well as those tested by other researchers. Comparisons of the results show that the proposed model predicts the measured results with an overall average error of 2% and a coefficient of variation of 1.48. This work forms an effective tool for investigation of inelastic behavior of a concrete and reinforcing bar composite.

### Introduction

Great deal of research effort in the field of structural engineering has been devoted toward prediction of the actual behavior of reinforced concrete elements. Lately, with the help of computers, large emphasis has been directed towards the use of finite element method. Many concrete structures such as beams, walls, slabs, shear walls, and shells are simulated as being in a state of plane stress.

In an effort to improve and facilitate the use of the finite element procedure different constitutive models were developed [1-5]. The success of such analysis is greatly influenced by the accuracy of modeling the composite material behavior.

This paper focuses on adapting the existing linear elastic algorithms [6] to incorporate the non-linear behavior of the composite material using the smeared crack

model [5]. The model uses a true stress-strain relationship for concrete under compression, considers the effect of tension stiffening after cracking (TSAC), and accounts for the possibility of developing single or double crack conditions.

### Constitutive Model

#### Composite behavior up to cracking

The composite material considered herein is constituted of concrete and reinforcing steel. The behavior of the concrete, up to cracking is assumed to be isotropic and linearly elastic. Thus, the concrete property matrix which relates the stresses and strains can be written, from Hooke's law for plane stress, as:

$$\{\sigma\}_c = [D]_c \{\varepsilon\}_c \quad (1)$$

in which

$$\{\sigma\}_c = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_{\text{concrete}}, \quad \{\varepsilon\}_c = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix}_{\text{concrete}} \quad \text{and} \quad [D]_c = \frac{E_c}{1-\nu^2} \begin{vmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{vmatrix}$$

where

$E_c$  = modulus of elasticity of concrete

$\nu$  = poisson's ratio of concrete

$D_c$  = uncracked property matrix of concrete

If, however, the concrete element contains reinforcement parallel to an arbitrarily oriented  $x'$ -axis, which makes an angle  $\theta$  with the  $x$ -axis then the reinforcement stress-strain relationship can be expressed as [7]:

$$\{\sigma\}'_c = [D]_c \{\varepsilon\}'_c \quad (2)$$

in which

$$\{\sigma\}'_s = \begin{Bmatrix} \sigma_x \\ 0 \\ 0 \end{Bmatrix}, \quad \{\varepsilon\}'_c = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad \text{and} \quad [D]'_s = \begin{bmatrix} \rho E_s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where  $\rho$  and  $E_s$  are, respectively, the reinforcement ratio and modulus of elasticity. Further, if the element contains reinforcement with  $N$  orientations then, assuming no slippage between concrete and reinforcement, the stiffness matrix of the compo-

site with respect to the global X and Y axes (Fig. 1) can be defined as:

$$[D]_{\text{comp.}} = [T]^T [D]_c [T] + \sum_{i=1}^N [T(\theta_i)]^T [D]_{si}' [T(\theta_i)] \quad (3)$$

where  $T$  is the transformation matrix [8], and  $\theta_i$  is the direction angle of the  $i$ th reinforcement with respect to the global X and Y axes (Fig. 1). It should be indicated here that the transformation matrix  $[T]$  will differ for each of the components.

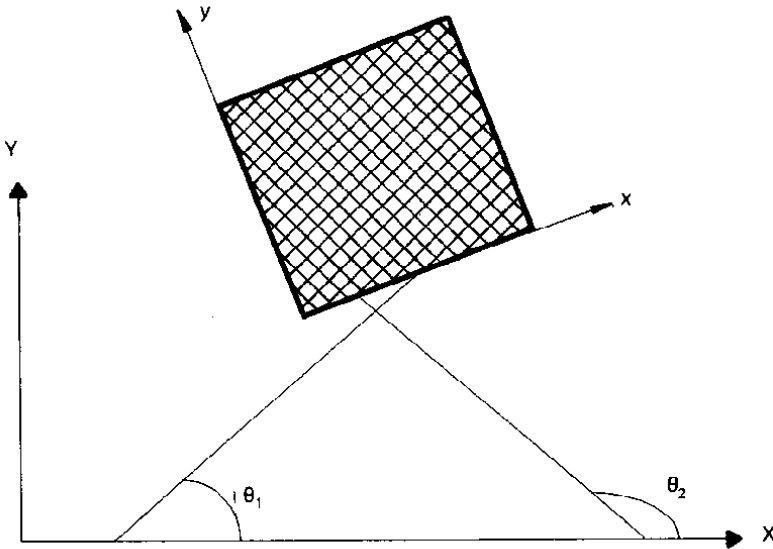


Fig. 1. Local vs. global axes and reinforcing steel orientations.

### Composite behavior after cracking

The use of linear elastic isotropic constitutive law as of Eq. 1 in constructing the stiffness matrices of concrete after cracking may not be realistic. This is so because as the principal tensile stress exceeds the tensile capacity of concrete, the concrete cracks and deviates from the linear elastic behavior. Also, as the principal compressive stress exceeds the proportional limit of concrete under compression, the modulus of elasticity of concrete decreases with the increase in the principal compressive strain. A more realistic constitutive law may be the one that reflects the influence of the strain variation on the moduli of elasticity. This may be arrived at as follows:

The principal strains  $\epsilon_1$  and  $\epsilon_2$  in two arbitrary orthogonal directions such as parallel and perpendicular to the crack direction, respectively (see Fig. 2) can be expressed as:

$$\epsilon_1 = \frac{\epsilon_x + \epsilon_y}{2} + \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \quad (4)$$

$$\epsilon_2 = \frac{\epsilon_x + \epsilon_y}{2} - \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \quad (5)$$

$$\text{and } \alpha = \frac{1}{2} \tan^{-1} \left( \frac{\gamma_{xy}}{\epsilon_x - \epsilon_y} \right) \quad (6)$$

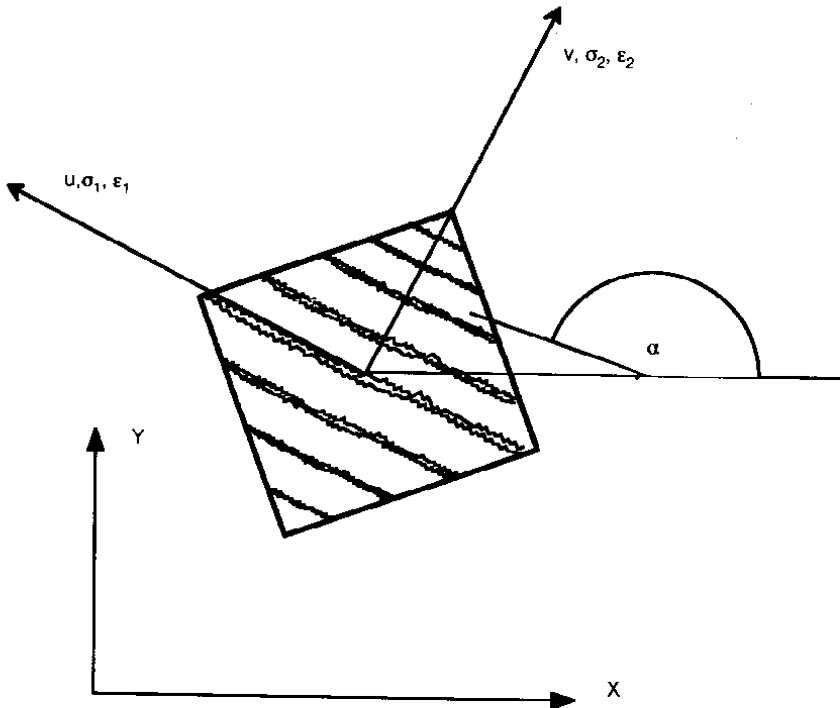


Fig. 2. Principal stresses, strains, and crack orientation.

where  $\alpha$  is the angle measured counterclockwise from the global X-axis to the axis of the crack, and  $\epsilon_x$ ,  $\epsilon_y$ , and  $\gamma_{xy}$  are, respectively, the normal and shear strains with reference to the global X and Y directions.

Stresses carried by concrete and reinforcement can be determined from the above computed principal strains,  $\varepsilon_1$  and  $\varepsilon_2$ . Thus, the principal tensile stress,  $F_t$ , carried by concrete, is:

$$F_t = \varepsilon_1 E_1 \quad \text{for } \varepsilon_1 < \varepsilon_{cr} \quad (7)$$

where  $E_1$  is the initial modulus of elasticity of concrete which is equal to twice the ratio of the ultimate compressive strength of concrete,  $f'_c$ , to the corresponding strain,  $\varepsilon_0$  ( $E_1 = 2f'_c/\varepsilon_0$ ) and  $\varepsilon_{cr}$  is the cracking strain.

To account for the tension stiffening after cracking, the formula suggested by Vecchio and Collins [9] is adopted in this study. This is given by:

$$F_t = f_{cr} / \left(1 + \sqrt{200\varepsilon_1}\right) \leq F_s \quad (8)$$

where  $f_{cr}$  is the stress which corresponds to  $\varepsilon_{cr}$  and  $F_s$  is the upper bound on concrete TSAC which for  $N$  layers of steel in concrete is given by:

$$F_s \leq \sum_{i=1}^N \rho_i (f_{yi} - f_{si}) \cos^2 \phi_i \quad (9)$$

in which  $\rho$  is the ratio of  $i$ th steel layer in the direction of  $\phi_i$ , a counterclockwise angle for the  $i$ th steel measured from the  $i$ th steel layer to the direction normal to the crack. The stresses  $f_{yi}$  and  $f_{si}$  are, respectively, the yield stress and the stress in the steel compatible with the principal strain  $\varepsilon_1$ .

The secant modulus of elasticity for concrete in tension may be written as:

$$F_t = F_t / \varepsilon_1 \quad (10)$$

The principal concrete compressive stress corresponding to the strain  $\varepsilon_2$ , given by Carreira and Kuang-Han [10], can be written as:

$$F_c = f_{c \max} \frac{\lambda(\varepsilon_2 / \varepsilon_0)}{\lambda - 1 + (\varepsilon_2 / \varepsilon_0)^\lambda} \quad (11)$$

in which  $f_{c \max}$  is the maximum compressive strength for concrete in a direction parallel to the crack which, as a result of degrading effect caused by the crack on the orthogonal direction, is equal to or less than the ultimate compressive strength of concrete,  $f'_c$ , under uniaxial compression, given by Steven et al. [4], as:

$$f_{c \max} = f'_c / [0.8 + 0.34 (\varepsilon_1 / \varepsilon_0)] \quad (12)$$

The parameter  $\lambda$  is,

$$\lambda = [f_{e\max}/32.0]^3 + 1.55 \quad (13)$$

and  $\epsilon_0$  is the strain corresponding to ultimate compressive stress (see Fig. 3-c), given by Almussalam and Alsayed [11], as:

$$\epsilon_0 = [0.2 f_c' + 13.06] 10^{-4} \quad (14)$$

where,  $f_{c\max}$  and  $f_c$  are in MPa

Thus the secant modulus for concrete in compression corresponding to the principal strain  $\epsilon_2$  may be calculated as:

$$E_c = f_c / \epsilon_2 \quad (15)$$

The shear modulus for the cracked concrete can be approximated by [2]

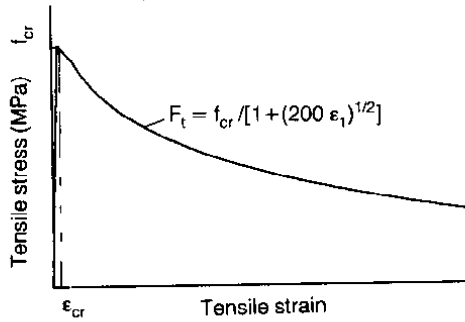
$$G = E_t E_c / (E_t + E_c)$$

Thus the cracked concrete stiffness matrix can be constructed as:

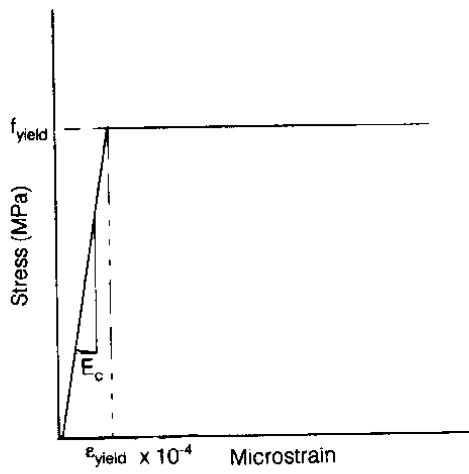
$$[D]_{cr} = \begin{bmatrix} E_t & 0 & 0 \\ 0 & E_c & 0 \\ 0 & 0 & G \end{bmatrix} \quad (17)$$

It can be noticed in Eq. 17 that, due to the lack of interaction of the cracked concrete in the two orthogonal directions, the Poisson's effect is neglected. It is also worth indicating here that the composite element may be in a state of biaxial tension (double crack condition), biaxial compression (no crack condition), or combined tension-compression (single crack condition). The stiffness matrix shown in Eq. 17 is for the element when it is in a state of combined tension-compression state. The ones for other states of stresses can be constructed in a likewise manner. To avoid the singularity of the constitutive matrix that may arise, in some cases, when the element is under biaxial tension, the shear modulus, for that particular case can be given a value of  $\beta E_t$ , where  $\beta$  is the shear interlock factor [1-5].

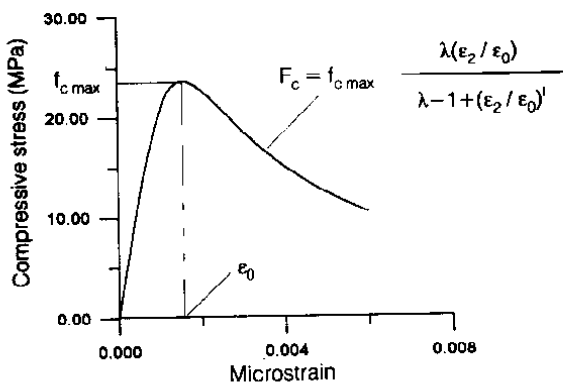
Now, the cracked concrete stiffness matrix of the composite with respect to the global X and Y axes can be formulated using the transformation matrices with the help of Eq. 3.



(a) Concrete stiffening after cracking



(b) Idealized stress-strain relationship for steel



(c) Stress-strain diagram for concrete under uniaxial compression stress.

**Fig. 3. Constitutive models for concrete and steel.**

### Finite Element Formulation

The element stiffness matrix,  $[k]$ , of the triangular element, which is the element considered in this study, can be evaluated as:

$$[k] = [B]^T [D]_{\text{comp}} [B] \quad (18)$$

where  $[B]$  is the matrix relating the assumed displacement field parameters to joint displacement [8]. From there on, the global stiffness matrix  $[K]$  of the whole structure can also be constructed [12]. Thus the force-displacement relationship is obtained:

$$\{P\} = [K] \{U\} \quad (19)$$

where  $\{P\}$  is the nodal forces vector (known values), and  $\{U\}$  is nodal displacements vector (unknown values). The algorithm suggested is shown in Fig. 4.

### Experimental Verification

In order to verify the computer model suggested here, the analytical and experimental responses of reinforced concrete panels subjected to in-plane stresses are compared. Comparison is made with the results of specimens that were tested as part of this study program as well as those tested by others.

### Test Program

Four panels were cast and tested in flexure. All panels were  $20 \times 300$  mm in cross-section and 600 mm in length. The test span was 500 mm. Two specimens were reinforced with two layers of woven steel mesh laid at  $45^\circ$  and  $135^\circ$  to the global coordinates. The other two were reinforced with one orthogonal layer of square mesh and one layer of woven mesh inclined at  $45^\circ$  to the global X-axis. The specimens were simply supported and subjected to two equal concentrated loads, symmetrically placed about mid-span and were 100 mm apart.

### Material properties

The mix proportions considered in this study were (by weight) 1: 1.5: 0.45 (cement: sand: water). Ordinary Portland cement and natural desert sand were used as concrete constituents. The average concrete compressive strength, for 50 mm-cube, was 61 MPa, modulus of rupture was 0.91 MPa, and Poisson's ratio was 0.16.

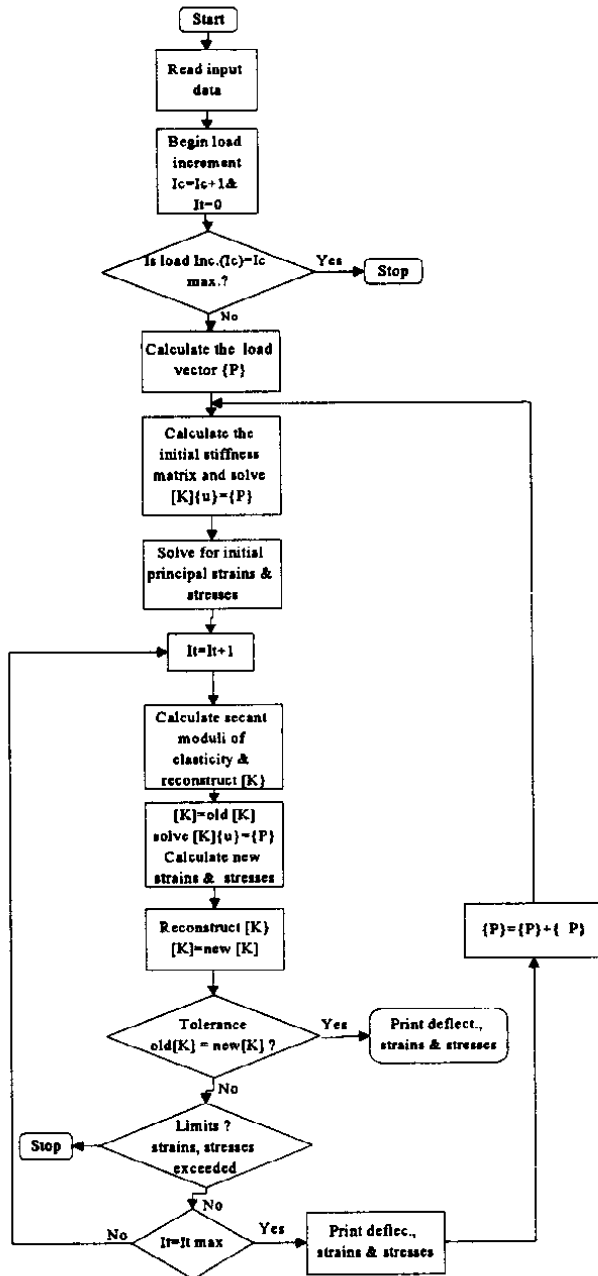
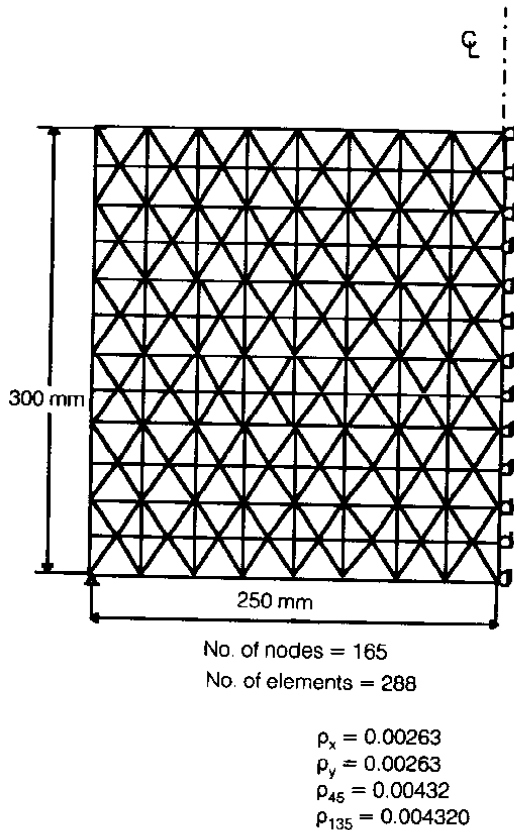


Fig. 4. Non-linear algorithm flow chart .

The spacing, diameter, and yield strength of the welded square mesh were, respectively, 12.7 mm, 0.923 mm, and 112 MPa. The corresponding values of the woven mesh were, respectively, 6.35 mm, 0.833 mm and 175 MPa. The Poisson's ratio for the square and woven steel mesh was 0.2. The shear interlock factor was assumed to be equal to 0.25 [5].

### Comparison of the results

A typical finite element mesh for the test specimen as used in the analysis is shown in Fig. 5. The average measured load deflection diagram for different mesh configurations along with the corresponding analytical results are shown in Figl 6. The average error in the ratio of the difference between the experimental and the



**Fig. 5. Finite element mesh and reinforcement ratios for test specimen (Reinforced with one layer of square mesh and one layer of woven mesh) .**

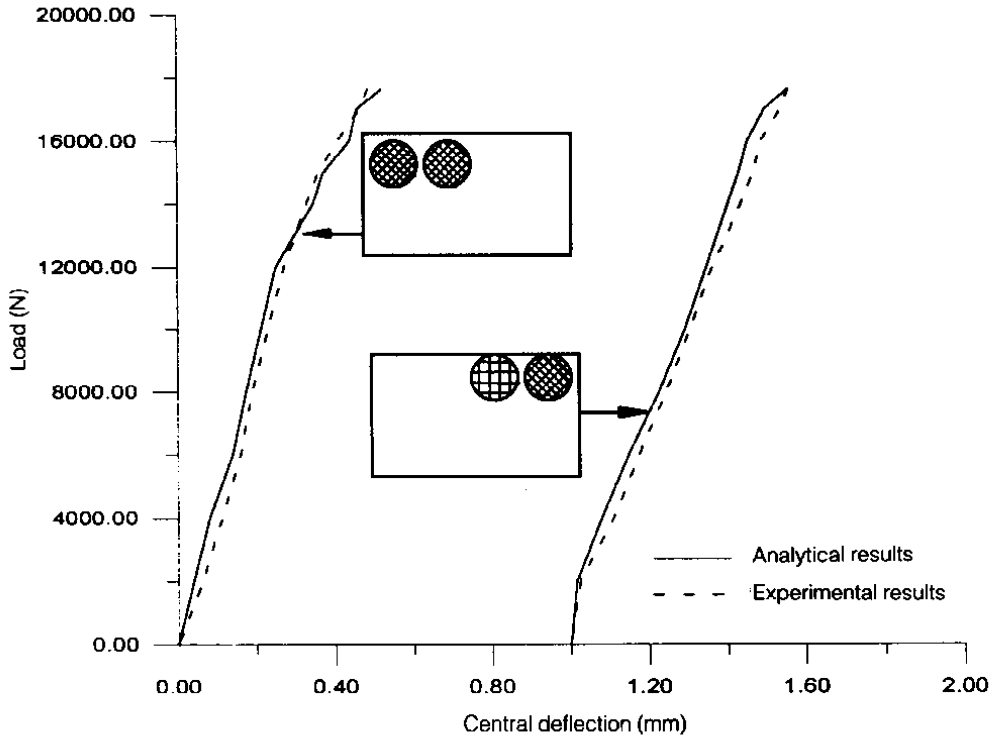


Fig. 6. Comparison between analytical and experimental results (current study) .

measured results to the measured results (EREM) is 4.6% and the corresponding coefficient of variation (CCOV) is 1.72. This clearly shows that there is a good agreement between the experimental and the analytical results. This is true for the load-deflection behavior as well as the prediction of the ultimate strength.

### Further Verification

The Validity of the model is also checked against the load-deflection response of beam A-1 tested by Bresler and Scordelis [13]. The experimental and analytical load-deflection curves are shown in Fig. 7. The average EREM is 2.03% and the CCOV is 4.51. This indicates that the model provided an excellent prediction to the measured values.

Furthermore, the comparison between the measured and predicted results for beams B-N-9 and B-1-11 tested by Al-Shaikh and Al-Zaid [15] is shown in Fig. 8. The average EREM for the two beams is 1.5% and the CCOV is 2.74.

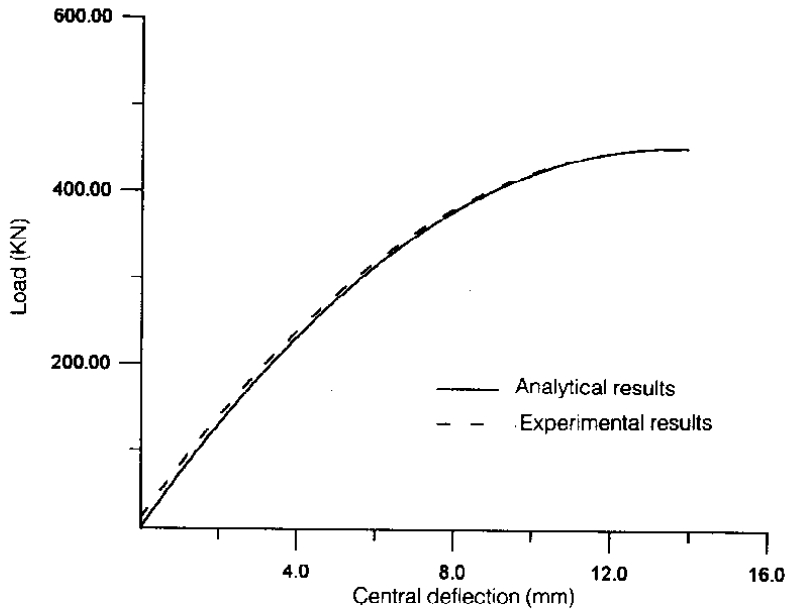


Fig. 7. Comparison between analytical and experimental results (beam A-1, Ref. 13).

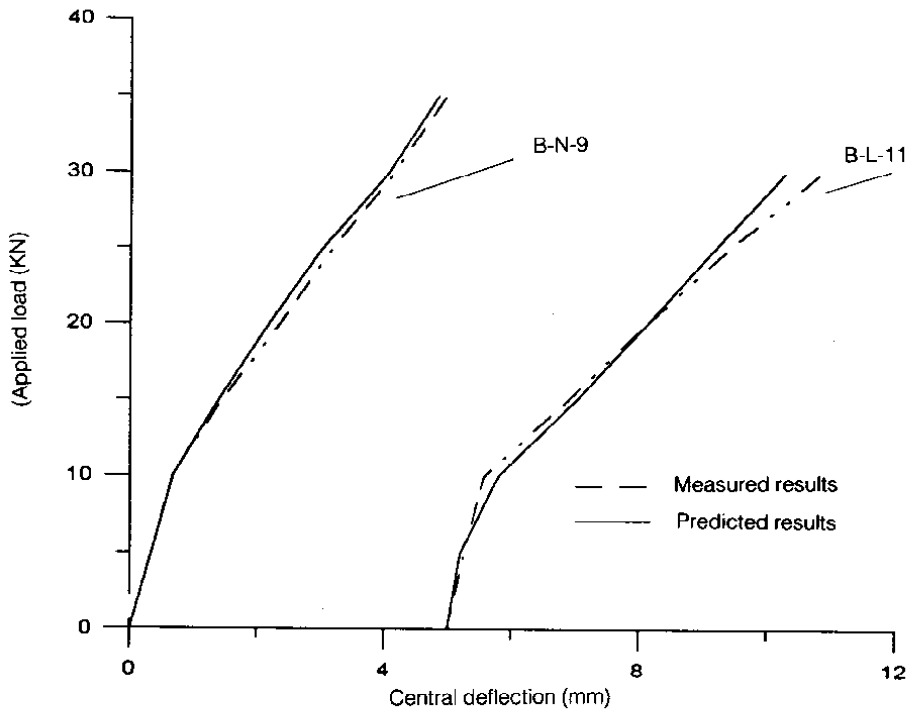


Fig. 8. Predicted vs. measured load deflection curve (Beams: B-N-9 and B-L-11; Ref. 15).

The overall average EREM of all specimens considered in the comparison between the experimental and the analytical results is 2% and the CCOV is 1.48. These close agreement between the measured and predicted results may be regarded as an evidence of the validity of the computer model that was developed in this study to predict the non-linear behavior of the composite structures under in-plane loads.

It is of interest to point out, not shown here, that to study the effect of TSAC on the overall prediction accuracy of the suggested model, prediction of the experimental results with and without considering TSAC was carried out. It was found that including the effect of TSAC highly improved the accuracy of the prediction. This was confirmed previously by other researchers [1,2]. Prediction of the experimental results was also attempted by replacing the shear modulus as given in Eq. 18 with a constant reduced shear modulus ( $\beta E_p/2$ ) as suggested by Hu and Schnobrich [1] and Hand *et. al.* [14]. It was observed that when the applied stress approached its ultimate value the prediction accuracy had an appreciable dependency on the value of the chosen  $\beta$ . This contradicts earlier founding [14] where it was reported that the prediction accuracy was insensitive to the value of the selected  $\beta$  (between 0 and 1).

Similar work was performed to investigate the impact of including or not the degradation of strength parallel to crack orientation and also the possibility of developing double crack condition. It was found that incorporating the influence of these two parameters helped in reducing the difference between the predicted and measured values.

### Summary and Conclusions

This study presented a simplified nonlinear iterative finite element model, implemented a 486 personal computer. The suggested algorithm is based on the smeared crack approach and uses a realistic stress-strain relationship for the composite constituents. It also considers the influence of concrete stiffening after cracking, stress degrading parallel to the crack, and possibility of developing single or double crack condition.

The accuracy of the model in predicting the experimental results was checked by comparing the predicted values with the corresponding values measured by the author as well as those measured by other investigators. Results of the comparisons showed that the predicted values are in good agreement with the measured ones. Based on the results of this work, the following conclusions may be drawn:

1. The model suggested herein provides an easy and yet accurate tool to predict the nonlinear behavior of the reinforced concrete elements subjected to in-plane loads. Data needed to utilize the model are the engineering properties of the reinforced concrete constituents, structural properties, and joint and/or distributed load data.
2. Including the effect of concrete stiffening after cracking, degrading effect in compression capacity parallel to the crack orientation, and the possibility of developing double crack condition greatly improves the accuracy of predicting the actual nonlinear behavior.
3. Using a constant value of reduced shear modulus helps in overcoming the numerical difficulty that may be encountered when the elements surrounding one node develops cracks in the same direction. However, using one value of shear interlock factor for all load history appears to be unrealistic choice.

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## التحليل غير المرن للأعضاء الخرسانية المسلحة بطريقة الأجزاء المتناهية تحت تأثير الأحمال المستوية

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(استلم في ١٩٩٣/٦/٩ م؛ وقبل للنشر في ١٩٩٤/١٠/٢٤ م)

ملخص البحث . هذه الورقة تقدم حلاً تكرارياً وغير مرن باستخدام طريقة الأجزاء المتناهية والحاسوب الشخصي ٤٨٦ للتنبؤ بشكل ويتمدد التشققات في الأعضاء الخرسانية المسلحة الواقعة تحت تأثير أحمال مستوية . النموذج المقترح مبني على تمثيل التشققات الدقيقة المتقاربة ومحاكاة حالة الإجهاد في مستوى واحد، ويأخذ في الاعتبار تأثير صلابة الخرسانة في منطقة الشد بعد تشققها وكذلك تناقص قوة مقاومة الضغط نتيجة التشقق مع اعتبار ظروف التشققات الأحادية والثنائية .

ولقد تم فحص صلاحية النموذج بمقارنة نتائجه مع نتائج الاختبارات لعينات تم اختبارها من قبل المؤلف وأخرى من قبل باحثين آخرين . وقد أوضحت نتائج المقارنات بأن النموذج المقترح تمكن من محاكاة نتائج الاختبارات بخطأ بلغ متوسطه العام ٢٪ ومعامل تغير قدره ٤٨ ، ١ . إن هذا الأنموذج يعتبر أداة فعالة للبحث في الأداء غير المرن للعناصر المركبة من الخرسانة وحديد التسليح .