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Test for Monotone Variance Residual Life Class of Life Distributions Based on Total Time on Test Transformation

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Abstract. Test for exponentially against decreasing (increasing) variance remaining life distribution DVRL (IVRL) is introduced. We construct test statistic based on scaled total time on test (TTT)-transform, to test exponentiality versus DVRL (IVRL) class. The distribution of the statistic is investigated via simulation. An example of 40 patients suffering from blood cancer disease (leukemia) is considered as a practical application of the proposed test in the medical sciences.

Keywords: Decreasing (Increasing) variance remaining life distribution DVRL (IVRL), Exponentiality, Total time on test (TTT-test).

1. Introduction

Reliability has recently been very important in many aspects of real life situations. Initially, reliability was motivated by its application in the filed of military and space technology, but now has become very important in production planning, design, operation and maintenance modeling.

Lauer [1], Gupta [2] and Gupta *et al*. [3] studied the characterization of DVRL and used it to find better bounds on moments and survival function. Gupta and Kirmani [4] charactrized the distribution to the univariate and the bivariate cases. Testing exponentiality versus (IFR, IFRA, NBU, NBUE, DMRL) classes have got a good deal of attention in the literature. For this literature, we refer the reader to the surveys by Doksum and Yandell [5], Barlow and Proschan [6], Hendi and Abuoummah [7], Kanjo [8], Alwasel [9] and Abu-Youssef [10] among others.

Let T denote the life time of an equipment distribution function $F(t) = P(T \le t)$, survival function $\overline{F} = 1 - F$. We assume that the density $f(t)$ can be obtained by differentiating $F(t)$. The failure rate $r(t) = f(t)/\overline{F}(t)$. The mean life $\mu = \int_0^\infty \overline{F}(u)du$ and the variance life $\sigma^2 = \text{var}(T)$ both assumed finite.The mean residual life (MRL) and the variance residual life are defined as:

$$\mu(t) = E\{T - t \mid T \ge t\} = \frac{\int_t^\infty \overline{F}(u)du}{\overline{F}(t)}, \qquad t \ge 0,$$

and

$$\sigma^2(t) = \text{var}\{T - t \mid T \ge t\} = \text{var}\{T \mid T \ge t\} = E[T^2 \mid t] - \mu^2(t).$$

Consider

$$E[T^2 \mid t] = -\int_0^\infty u^2 d\overline{F}(u/t) \tag{1}$$

by integrating by parts we have:

$$\sigma^2(t) = \frac{2}{\overline{F}(t)} \int_0^\infty \int_y^\infty \overline{F}(x)dxdy - \mu^2(t).$$

A distribution function F is said to be a decreasing (increasing)variance residual life DVRL (IVRL) if $\sigma^2(t)$ is nonincreasing (nondecreasing)function of t. Differentiating Eq. (1) with respect to t, we obtain:

$$\frac{d\sigma^2(t)}{dt} = r(t)[\sigma^2(t) - \mu^2(t)]$$

Since $\overline{F}(t)$ is DVRL (IVRL) then, $\sigma^2(t) \le (\ge) \mu^2(t)$ implies $\sigma^2(t) + \mu^2(t) \le (\ge) 2\mu^2(t)$, hence:

$$\frac{2}{\overline{F}(t)} \int_t^\infty \nu(y)dy \le (\ge) 2\mu^2(t).$$

Now, we have the following definition.

1.1. Definition

A life distribution F, with $F(0) = 0$ and its survival function $\overline{F}$ are said to have DVRL (IVRL) if :

$$\overline{F}(t)\int_t^\infty\int_y^\infty \overline{F}(x)dxdy \le (\ge)(\int_t^\infty \overline{F}(u)du)^2, \tag{2}$$

The main theme of this paper is dealing with the problem of testing $H_0 \ : \ \overline{F} = \exp(-\lambda t), \lambda > 0$, versus $H_1 \ : \ \overline{F}$ is DVRL (IVRL) using Eq. (2). In section 2 of this paper, we give a brief account of TTT-transform and TTT-plot and present a characterization for class of life distribution DVRL (IVRL) via the scaled total time on test. In section 3, we derive the empirical test statistic for the DVRL (IVRL) based on the scaled TTT-transforms. In Section 4, a small sample study of this test statistics is performed through simulation. The power estimates of this statistic are given in section 5, with respect to some commonly used distribution in reliability. We conclude this paper with an application in section 6.

2. The Idea of Total Time on Test (TTT-test)

Let F be the life distribution with survival function $\overline{F} = 1 - F$ and finite mean $\mu = \int_0^\infty \overline{F}(u)du$. We present the following definition of Barlow and Campo [11].

2.1. Definition

(i) The TTT-transform $H^{-1}(t)$ of F where $t = F(x)$ is defined by:

$$H^{-1}(t) = \int_0^{F^{-1}(t)} \overline{F}(u)du$$

where $F^{-1}(t) = \inf\{x \ : \ F(x) \ge t\}$ It is easy to verify that the mean μ, of F is given by:

$$H^{-1}(1) = \int_0^{F^{-1}(1)} \overline{F}(u)du$$

(ii) The function

$$\phi_F(t) = H^{-1}(t) / H^{-1}(1) \tag{3}$$

is called the scaled TTT-transform. Note that if F is the exponential distribution then TTT-transform is given by $\phi_F(t) = \int_0^{F^{-1}(t)} e^{-\lambda u} du / \int_0^{F^{-1}(1)} e^{-\lambda u} du = t$ for $0 \le t \le 1$.

Now, let $t_{(0)} < t_{(2)} < \ldots \ t_{(n)}$ to be an ordered sample from a life distribution F, where $t_0 = 0$ and let

$$D_j = (n - j + 1)(t_{(j)} - t_{(j-1)}), \ j = 1, 2, ..., n$$

then

$$S_j = \sum_{k=1}^{j} D_k = t_{(1)} + t_{(2)} + \ldots + t_{(j)}$$

denote the TTT-transform at $t_{(j)}$, where $S_0 = 0$. The value $W_j = S_j / S_n$ is an estimate of the scaled TTT-transform. An estimate of scaled TTT-transform is known as empirical scaled TTT-transform and is obtained as:

$$\phi_F\left(\frac{j-1}{n}\right) = W_{j-1},\ j = 1, 2, \ldots, n$$

The TTT-plot is obtained by plotting W_{j-1} against $\frac{j-1}{n}$, for $j = 1, 2, \ldots, n$, i.e. $(W_{j-1}, \frac{j-1}{n})$, for $j = 1, 2, \ldots, n$ and connecting the plotted points by straight lines. It has been shown, Barlow and Doksum [12, p. 237], using Glivevenko-Cantelli Lemma, that for strictly increasing F, W_j converges to $\phi_F(t)$ with probability one and uniformly in [0,1] as n tends ∞, and j/n converge to t. Scaled TTT-transform for some families of life distribution are given by Barlow and Compo [11], Barlow [13], Bergman [14] and Klefsjo [15, 16].

2.2. Theorem

Let F and $\phi_F(t)$ be as in (2.1), then we have the following: F is DVRL (IVRL) if:

$$\overline{F}(t)\int_{F(t)}^{1}(1-\phi_F(s))\phi_F'(s)(1-s)^{-1}ds \le (\ge)[1-\phi_F(s)]^2 \tag{4}$$

Proof: We prove the theorem for basic class, while the proof for the dual class can be carried out in parallel steps. From Eq. (2), the life distribution has DVRL property, if:

$$\overline{F}(t)\int_t^\infty\int_x^\infty \overline{F}(u)du dx \le [\int_t^\infty \overline{F}(u)du]^2$$

Rearranging the terms and using the transformation $x = F^{-1}(s)$ i.e., $f(x)dx = ds$ or $dx = ds/[f(F^{-1}(s))]$, yields

$$\overline{F}(t)\mu_F\int_{F^{-1}(t)}^{1}(1-\mu_F^{-1}\int_0^{F^{-1}(s)}\overline{F}(u)du)[f(F^{-1}(s)]ds \le$$

$$[\mu_F(1-\mu_F^{-1}\int_0^{F^{-1}(s)}\overline{F}(u)du)]^2$$

$$\overline{F}(t)\mu_F\int_{F(t)}^{1}(1-\phi_F(s))\mu_F\phi_F'(s)(1-s)^{-1}ds \le \mu_F^2[1-\phi_F(s)]^2 \tag{5}$$

This completes the proof.

3. Test Statistic Based on the Scaled TTT-Transform

In this section, we present a test statistic using the scaled TTT-transform for testing exponentially or H_0 against H_1 or DVRL (IVRL) class (not exponential) based on a sample $t_{(0)}, t_{(2)}, \ldots t_{(n)}$ from F. Now, since the TTT-plot W_{j-1} converges to the scaled TTT-transform $\phi_{F(t)}$, for $0 \le t \le 1$ as $n \to \infty$ and $\frac{j-1}{n} \to t$, then TTT -plot based on an ordered sample

$$0 \le t_{(0)} \le t_{(2)} \le \ldots t_{(n)}$$

behave as $\phi_F(t)$ does. This suggests the following test statistic based on the scales TTT-transform. Since F is DVRL (IVRL), then from Eq. (4), let:

$$U(F(t)) = [1-\phi_F(s)]^2 - \overline{F}(t)\int_{F(t)}^{1}(1-\phi_F(s))\phi_F'(s)(1-s)^{-1}ds \tag{6}$$

for $0 \le s \le 1, 0 \le t \le 1$. Integrating both sides of the above equation over [0,1] with respect to $z = F(t)$, we get:

$$\Delta_F = \int_0^1 [1-\phi_F(s)]^2 dz - (1-z)\int_0^1 (\int_z^1 (1-\phi_F(s))\phi_F'(s)(1-s)^{-1} dsdz$$

By using Eq. (7) we estimate Δ_F at a specified time t, as follow, let:

$$\begin{aligned}\hat{\Delta}_{F_n} = \sum_{i=1}^{n}\sum_{j=i}^{n}\{(1-w_{(j-1)})^2 - (n-i+1)(1-w_{(j-1)})\\ (w_{(j-1)} - w_{(j-2)})(n-j+1)^{-1}(t_{(j)} - t_{(j-1)})\}(t_{(i)} - t_{(i-1)})\end{aligned} \tag{7}$$

To reduce the size of the test statistic we use:

$$\hat{\Delta}_F^* = \hat{\Delta}_{F_n} / n \tag{8}$$

where $i = 1, 2, \ldots, n, j = 1, 2, \ldots, n, t_{(1)}, t_{(2)}, \ldots, t_{(n)}$ are the ordered statistics of the independent random sample $t_1, t_2, \ldots, t_n$, and $t_{(0)} = 0$. Note that $H_0 : \Delta_F = 0$ if F is exponential $H_1 : \Delta_F < (>) 0$ if F is DVRL (IVRL) and not exponential using the inequality (1.8).

4. Simulation of Small Samples

We calculate, via MonteCarlo Method, the empirical critical points of $\hat{\Delta}_n$ in Eq. (8) for samples 5(1)40. Tables 1 gives the lower and the upper percentile points for 1%, 5%, 10%, 90%, 95%, 98% and 99%. The calculations are based on 10000 simulated samples of sizes $n =$ 5(1)40.

Note that the value of critical values of percentiles are decreasing when the sample sizes are increasing and are also symmetric about the 50% and this implies the estimator $\hat{\Delta}_{F_n}$ is asymptotic normal.

Table 1. Cirtical values $\hat{\Delta}_n$

n	1%	5%	10%	90%	95%	98%	99%
5	.0006	.0588	.0810	.2491	.3017	.3739	.4500
6	.0025	.0623	.0803	.2229	.2590	.3201	.3646
7	.0063	.0671	.0819	.2087	.2383	.2802	.3112
8	.0160	.0673	.0832	.2010	.2261	.2641	.2986
9	.0376	.0702	.0835	.1910	.2142	.2440	.2717
10	.0421	.0707	.0829	.1870	.2078	.2338	.2551
11	.0493	.0719	.0832	.1833	.2014	.2246	.2437
12	.0513	.0723	.0830	.1786	.1959	.2208	.2382
13	.0543	.0733	.0833	.1759	.1927	.2125	.2270
14	.0572	.0749	.0855	.1743	.1893	.2090	.2247
15	.0587	.0757	.0848	.1712	.1864	.2060	.2187
16	.0590	.0765	.0856	.1698	.1844	.2022	.2133
17	.0597	.0757	.0854	.1669	.1816	.1983	.2106
18	.0618	.0763	.0859	.1653	.1790	.1954	.2073
19	.0628	.0777	.0859	.1634	.1770	.1941	.2065
20	.0632	.0775	.0864	.1624	.1763	.1928	.2049
21	.0629	.0774	.0857	.1615	.1733	.1878	.1977
22	.0637	.0779	.0865	.1597	.1726	.1874	.1977
23	.0657	.0793	.0871	.1593	.1724	.1884	.1995
24	.0653	.0793	.0873	.1580	.1706	.1865	.1974
25	.0671	.0798	.0875	.1580	.1692	.1834	.1919
26	.0663	.0805	.0877	.1558	.1676	.1802	.1909
27	.0691	.0808	.0881	.1559	.1681	.1833	.1921
28	.0675	.0815	.0884	.1543	.1653	.1792	.1886
29	.0691	.0810	.0884	.1541	.1655	.1775	.1861
30	.0698	.0819	.0890	.1527	.1638	.1772	.1840
31	.0705	.0821	.0891	.1526	.1641	.1760	.1841
32	.0703	.0820	.0893	.1511	.1624	.1753	.1849
33	.0707	.0828	.0896	.1503	.1619	.1744	.1837
34	.0711	.0831	.0897	.1503	.1607	.1726	.1804
35	.0712	.0826	.0893	.1495	.1597	.1718	.1789
36	.0717	.0836	.0902	.1480	.1582	.1710	.1785
37	.0708	.0834	.0897	.1493	.1587	.1694	.1778
38	.0717	.0842	.0909	.1477	.1573	.1693	.1759
39	.0724	.0841	.0906	.1473	.1566	.1680	.1764
40	.0725	.0847	.0908	.1469	.1567	.1678	.1745

5. The Power Estimates

The power of the test statistics $\hat{\Delta}_n$ is considered for 5% percentile in Table 2 for three of the most commonly used alternatives (see Hollander and Proschan [17]). They are:

(i) Linear failure rate : $\bar{F}_\theta = e^{-x-\frac{\theta x^2}{2}}$, $x > 0, \theta > 0$

(ii) Makeham : $\bar{F}_\theta = e^{-x-\theta(x+e^{-x}-1)}$, $x \geq 0, \theta > 0$

(iii) Weibull : $\bar{F}_\theta = e^{-x^\theta}$, $x > 0, \theta > 0$

These distributions are reduced to exponential distribution for appropriate values of θ.

Table 2. Power Estimate of $\hat{\Delta}_n$

Distribution	θ	n=10	n=20	n=30
F_1	1	0.654	0.872	0.966
Linear failure	2	0.980	0.999	1.000
rate	3	1.000	1.000	1.000
F_2	1	0.482	0.685	0.853
Makham	2	0.818	0.963	0.997
	3	0.965	0.999	1.000
F_3	1	0.146	0.332	0.473
Weibull	2	0.710	0.999	1.000
	3	0.942	1.000	1.000

Note that the power estimates of the test $\hat{\Delta}_{v_n}$ are increasing when the parameter θ is far for from exponantial and when the size of the sample n is increasing.

6. Application

Consider the data in Abouammoh *et al.* [18]. These data represent 40 patients suffering from blood cancer from one of the Ministry of Health Hospitals in Saudi Arabia and the ordered life times (in days) are 115, 181, 255, 418, 441, 461, 516, 739, 743, 789, 807, 865, 924, 983, 1024, 1062, 1063, 1169, 1191, 1222, 1222, 1251, 1277, 1290, 1357, 1369, 1408, 1455, 1478, 1549, 1578, 1578, 1599, 1603, 1604, 1696, 1735, 1799, 1815 and 1852. Using Eq. (8), we find that the value of statistic $\hat{\Delta}^*_{F_n} = 0.0597$. This leads to accepting H_0 which states that this set of data has exponential property under significance level $\alpha = 0.05$.

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دراسة اختبار فصل توزيع تباين بقاء ا ياة باستخدام ويل الزمن الكلي

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ملخص البحث. سوف نقدم في هذ البحث اختبار التوزيع الأسي ضد توزيع متزايد (متناقص) لتوزيع تباين بقاء الحياة بالاعتماد على تحويل الزمن الكلي القياسي. تم فحص توزيع إحصاء الاختبار وعمل جدول للقيم الحرجة للمئينات للعينات الصغيرة باستخدام المحاكاة. وتم استخدام مثال لبيانات حقيقية لعينة من مرضى سرطان الدم والذين يبلغ عددهم ٤٠ مريضا لإبراز مدى تطبيق البحث في المجال الطبي.