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Approximation for Seven-gluon Scattering

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Abstract. In this paper we give an approximate expression for the squared matrix element of the process $gg \rightarrow ggggg$. This approximation is based on the infra-red reduction and is the result of six-gluon scattering, to leading order in the number of colors. It is shown using this approximation, that the error is dramatically reduced to about 1% compared to the exact result for seven gluon scattering.

We compared the analytic approximation form for the cross-section which we obtained with the exact form and we found that there is 99% agreement. We can use this analytic form to study the five jets.

We discovered that the factor which was suggested by Striling is independent of jet kinematics, the accuracy of the approimation will presumably be rather sensitive to kinematical cuts since the approximation which was presented by Striling will not be accurate event-by-event in phase space.

Calculations Technique

In the present there is no analytic expression for the squared matrix elements of the process $gg \rightarrow ggggg$ only existing expression for unsquared subamplitudes. These subamplitudes are written in terms of spinors [1].

The matrix elements squared for the seven gluons scattering are difficult to obtain in a compact form and the reasons behind that are: firstly, each subamplitude of ref. [1] has very long expression in terms of spinors. Secondly, in the matrix elements squared for each subamplitude required traces with (18), and (more), objects of the external gluons momenta need to evaluate in terms of kinematical invariants. In fact these traces appeared in m_1, m_2, m_3 and m_4 with helicity configuration (+++----), (++-+---), (++--+--) and (+-+-+--) respectively.

The purpose of this paper is to give approximation form for the process $gg \rightarrow ggggg$ which was based on the calculations of six-gluon scattering, to leading order in the number of color , and some reduction factor. This expression is useful in simulating five jet events. At the same time we improved the result which was obtained by Maxwell.

We shall write the six-gluon scattering matrix elements squared, $|M_6|^2$, to leading order in the number of color, as

$$|M_6|^2 = |M_6^{PT}|^2 + |M_6^{rest}|^2 \tag{1}$$

$|M_6^{PT}|^2$ is the contribution of (--++++), and permuted, helicity orderings which is given by the formula of Parke and Taylor [2].

$$|M_6^{PT}|^2 = g^8 N_c^4 \left(N_c^2 - 1\right) \sum_{i<j} (ij)^4 \sum_P \frac{1}{(12)(23)\ldots(61)} \tag{2}$$

(i,j) means the dot product of four momenta $(P_i.P_j)$. The matrix elements squared of the expression in equation (2) are summed over color and helicities, but a color and helicity averaging factor should be supplied to these expressions. P denotes a sum over the 60 distinct non-cyclic permutations of 1, 2, 3, 4, 5, 6 up to cyclic and reverse reorderings.

For the remaining helicity orderings we shall write, following ref. [3].

$$|M_6^{rest}|^2 = 2g^8 N_c^4 \left(N_c^2 - 1\right) \sum_{P_6} \left[\frac{1}{6} H_1(1..6) + H_2(1..6) + \frac{1}{2} H_3(1..6)\right] \tag{3}$$

H_1, H_2 and H_3 in equation (3) are understood as the contributions of the (+-+-+-), (++-+--) and (+++---) helicity combinations, respectively. P_6 means a sum over all 720 permutations of the external gluon momenta (1, 2, 3, 4, 5, 6). The expressions for H_1 and H_2 are summarized in Table1.

The poles which appeared in Table 1 read as the following:

$$T_1 = t_{123} S_{12} S_{23} S_{45} S_{56} \tag{4}$$

$$T_2 = \pi_+ T_1, T_3 = \pi_- T_1 \tag{5}$$

π_+ and π_- in equation (5) denote permutations of the external momenta according to the following rules

$$\pi_+ : (123456) \to (234561)$$

$$\pi_- : (123456) \to (345612)$$

$$S = S_{12} S_{23} S_{34} S_{45} S_{61} S_{56}$$

We used the following kinematical quantities.

$$t_{ijk} = S_{ij} + S_{ik} + S_{jk}$$

$$U_{ijkl} = S_{ij} S_{kl} - S_{ik} S_{jk} + S_{jk} S_{il}$$

$$t_{jkl} = S_{ij} S_{kl} - S_{jk} S_{il}$$

Table1. The expressions for H_1 and H_2 in terms of kinematical invariants. [4].

Poles	$H_1(123456)$	$H_2(123456)$
T_1^2	$\frac{3}{4}a_1^2 S_{13}^2 S_{46}^2$	$\frac{1}{4}b_1^2 S_{12}^2 S_{56}^2$
T_2^2	O	$\frac{1}{2}b_2^2 S_{24}^2 S_{56}^2$
$T_2 T_3$	$3\left[(a_4+a_5)^2+2a_5 t_{135}^2 S_{34}S_{16}+a_5^2\right]$	$\left[(b_4+b_5)^2+2t_{124}^2 b_5 S_{16}S_{34}+b_5^2\right]$
$T_1 T_3$	O	$S_{12}^2\left[(b_6+b_7)^2+b_7(3b_7+2b_6)\right.$ $\left.+t_{124}^2\left(U_{3546}^2-4S_{35}S_{45}S_{46}S_{36}\right)\right]$
$T_1 S$	$3t_{234}a_1 S_{13}S_{46}(a_6+2a_7)$	$b_1\beta_1 t_{234}S_{12}^2 S_{56}$
$T_2 S$	O	$b_2 S_{24}S_{56}\left[\beta_2 t_{345}S_{56}+\beta_3 t_{121}\right]$
S^2	$\left[3S_{34}S_{16}((a_6+a_7)(a_8+a_9)+\right.$ $a_7 a_9+t_{135}^2 S_{13}S_{46}a_{10})+$ $\frac{3}{2}t_{234}t_{345}S_{13}S_{46}a_1(a_4+2a_5)$ $\left.-\frac{3}{4}t_{123}^2 a_2 a_3 a_5\right]$	$\left\{S_{12}S_{23}S_{56}\left[\beta_1\beta_3-t_{124}b_4 u_{3564}\right.\right.$ $\left.-2t_{124}^2 U_{1246}S_{35}S_{34}S_{56}\right]+$ $S_{12}S_{34}S_{56}S_{16}\left[\frac{1}{2}\beta_1\beta_2+t_{124}^2 b_{12}\right]-$ $\frac{1}{4}t_{123}^2 b_2 b_3 b_5+$ $\frac{1}{2}t_{234}t_{345}S_{12}S_{56}b_1\beta_3+$ $\left.\frac{1}{2}t_{234}b_3 b_7(S_{12}t_{234}b_1+2t_{123}\beta_2)\right\}$

For H_1

$$
\begin{aligned}
a_1 &= 2(t_{135}t_{123}-S_{13}S_{46}) \\
a_2 &= \pi_+ a_1, a_3 = \pi_+ a_2 \\
a_4 &= -t_{135}(t_{135}t_{234}t_{345}-t_{135}S_{34}S_{16}-t_{345}S_{15}S_{24}-t_{234}S_{35}S_{26}) \\
a_5 &= -S_{15}S_{35}S_{26}S_{24},\ a_6 = \pi_+ a_4, a_7 = \pi_+ a_5 \\
a_8 &= \pi_+ a_6,\ a_9 = \pi_+ a_7 \\
a_{10} &= -\tfrac{1}{2}(t_{1526}t_{2345}+t_{1524}t_{2563}+t_{2315}t_{2465}+t_{2315}S_{26}S_{45} \\
&\quad +t_{2465}S_{12}S_{35}+t_{2614}S_{52}S_{53}+t_{4563}S_{12}S_{25}+S_{12}S_{35}S_{24}S_{56})
\end{aligned}
$$

For H_2

$$b_1 = \pi_j a_1, b_2 = \pi_i b_1,\ \ b_3 = \pi_r b_2$$
$$b_4 = \pi_1 a_4,\ \ b_5 = \pi_1 a_5$$
$$b_6 = -t_{124}\left(t_{123}S_{35} - t_{124}S_{45} + t_{345}S_{56}\right)$$
$$b_7 = S_{12}S_{56}S_{35}$$
$$b_8 = \pi_r b_6,\ \ b_9 = \pi_r b_6$$
$$b_{12} = \frac{1}{2}\left(t_{2346}S_{14}S_{35} + t_{1345}S_{24}S_{36} + t_{1634}S_{23}S_{45}\right.$$
$$+t_{1463}S_{34}S_{52} + t_{1643}S_{35}S_{24} + t_{2615}S_{34}^2 - S_{23}S_{15}S_{46}S_{34}$$
$$\left.-S_{26}S_{13}S_{34}S_{45} - 2S_{13}S_{46}S_{34}S_{25}\right)$$

π_j, π_i, π_r and π_1 are permutations acting on the external momenta according to the following rules

$$\pi_j = (123456) \to (132546)$$
$$\pi_i = (123456) \to (241356)$$
$$\pi_r = (123456) \to (654321)$$
$$\pi_1 = (123456) \to (154326)$$

$$\beta_1 = \left(b_6 + 2b_7\right),\ \ \beta_2 = \left(b_8 + 2b_9\right)$$
$$\beta_3 = \left(b_4 + 2b_5\right)$$

the expression for H_3 is given by

$$H_3(1\ldots6) = t_{123}^3\left(\frac{t_{123}S_{34}S_{16} + 2t_{234}S_{45}S_{12}}{t_{234}t_{345}S_{12}S_{23}S_{34}S_{45}S_{56}S_{61}}\right) + \frac{\left(t_{123}t_{234}t_{345} - 2t_{234}S_{45}S_{12}\right)^2}{t_{234}^2t_{345}^2S_{34}^2S_{16}^2} - \frac{4t_{123}^2}{t_{234}t_{345}S_{34}S_{16}}$$

Seven-gluon Results

Now we assume the collinear limit in which one particle from the final state goes parallel to one particle from the final state $f11\overline{f}$ then one has

$$\lim_{(ij)\to 0}(ij)\left|M_n\right|^2 = g^2P_{gg}(z)\left|M_{n-1}\right|^2 \tag{6}$$

where $P_{gg}(z)$ the Altarelli - Parisi splitting Kernel and given by the following relation

$$P_{gg}(Z) = N_c\left[Z^4 + (1-Z)^4 + 1\right] / Z(1-Z) \tag{7}$$

here $(ij) \to 0$, $i \to (1-Z)a$ and a =i+j, Z in equation (7) is the energy fraction, where

$$Z = E_i / \left(E_i + E_j\right)$$

We shall use the reduction technique [5] to approximate $|M_n|^2$ for some (n-1) jet configuration then one has

$$|M_n|^2 \doteq \frac{|M_n|^2}{\left|M_n^{PT}\right|^2} I_{IR} X \left|M_n^{PT}\right|^2 \tag{8}$$

where I_{IR} denotes that the ratio of amplitudes is evaluated at a 'nearby' configuration where two of the final jets i, j, having the smallest invariant mass, have been replaced by a collinear pair of jets in the direction of $P_i + P_j$ with energy fraction Z and (1-Z). The ratio in equation (8) contains no Kinematical poles it will be reasonably slow-varying.

As we noted that the Parke and Taylor formula does not reduce by this way this means that $Z^4 + (1-Z)^4$ and 1 have different coefficients. We shall define

$$R = \sum_{l,\ m \neq j} (lm)^4 \Big/ \sum_{l \neq i,\ j} (la)^4 \tag{9}$$

We rewrite equation (2) in this limit as

$$\operatorname*{Lim}_{(ij)\to 0} (ij)\left|M_n^{PT}\right|^2 = g^2 f(R,Z)\left|M_{n-1}^{PT}\right|^2 \tag{10}$$

for general n with

$$f(R,Z) = N_c\left[R + Z^4 + (1-Z)^4\right] / (R+1)Z(1-Z) \tag{11}$$

and hence the equation in (8) written as

$$|M_n|^2 \doteq F(R,Z)\frac{|M_{n-1}|^2}{\left|M_{n-1}^{PT}\right|^2} X \left|M_n^{PT}\right|^2 \tag{12}$$

with

$$F(R,Z) = (R+1)\left[Z^4 + (1-Z)^4 + 1\right] / \left[Z^4 + (1-Z)^4 + R\right] \tag{13}$$

For seven gluon scattering (n=7), then

$$\left|M_7\right|^2 \simeq F(R,Z)\frac{\left|M_6\right|^2}{\left|M_6^{PT}\right|^2} X \left|M_7^{PT}\right|^2 \tag{14}$$

where $\left|M_6\right|^2$ is the matrix element squared for six-gluon scattering, to leading order in the number of color. $\left|M_6^{PT}\right|^2$ is the Parke and Taylor formula for $gg \to gggg$ and given in equation (2).

The exact matrix element squared for $\left|M_7^{PT}\right|^2$ for the process $gg \to ggggg$ is written as

$$\begin{aligned}\left|M_7^{PT}\right|^2 &= g^{10} N_c^5\left(N_c^2-1\right)\sum_{i<j}(ij)^4\sum_{P_7}\frac{1}{(12)(23)(34)(45)(56)(67)(71)} \\ &+\frac{1}{8}g^{10}N_c^3\left(N_c^2-1\right)\left\{\sum_{i<j}S_{ij}^4\sum_{P_7}\frac{1}{4S}\left[\frac{15A(1234567)}{S_{13}S_{16}S_{46}S_{74}S_{75}S_{53}}\right.\right. \\ &+\frac{12B(1234567)}{S_{24}S_{46}S_{36}S_{35}S_{57}S_{45}}+\frac{15C(1234567)}{S_{14}S_{42}S_{25}S_{53}S_{36}S_{13}} \\ &\left.\left.-\frac{2D(1234567)}{S_{14}S_{47}S_{73}S_{36}S_{62}S_{25}S_{54}S_{51}}\right]\right\}\end{aligned} \tag{15}$$

the functions A and B are given in Table 2 and the functions C and D are given in table 3. The functions X, Y and Z which are appeared in tables 2 and 3 are defined as

$$\begin{aligned}X(1..7) &= U_{7431}U_{5631}-2t_{156347}S_{13} \\ Y(1..7) &= U_{1456}U_{1457}-2U_{1756}S_{14}S_{45} \\ Z(1...7) &= t_{1457}S_{32}S_{26}+t_{1564}S_{12}S_{32}+t_{2756}S_{13}S_{24}+t_{2473}S_{15}S_{26} \\ &+t_{1275}S_{24}S_{36}+t_{3567}S_{12}S_{24}+t_{1627}S_{24}S_{35}+t_{1463}S_{27}S_{52}+t_{3764}S_{12}S_{52} \\ &+t_{1362}S_{47}S_{52}+t_{1734}S_{26}S_{52}+U_{6745}S_{12}S_{32}\end{aligned}$$

S in equation(15) reads as

$$S = S_{12}S_{23}S_{34}S_{45}S_{56}S_{67}S_{71}$$

P_7 in equation (15) means a sum over the 360 distinct non-cyclic permutations of 1, 2, 3, 4, 5, 6, 7 up to cyclic and reverse reorderings.

Table 2. The coefficients of A and B in the Parke and Taylor Formula for $gg \rightarrow ggggg$

A (1234567)	B (1234567)
$-[U_{7546}U_{7431}U_{3561}$	$U_{2453}U_{4675}U_{4563}$
$+U_{7546}X(1234567)$	$+2U_{2453}Y(4215673)$
$-U_{7431}X(5264317)$	$+U_{4563}X(5142763)$
$-U_{3561}X(4271563)]$	

Table 3. The coefficents of C and D in the Parke and Taylor formula for seven gluon scattering.

C (1234567)	D (1234567)
$U_{1453}U_{1342}U_{2563}$	$-\{3U_{1452}U_{4715}U_{3762}U_{4563}$
$+U_{2563}Y(1673425)$	$-U_{4563}U_{3762}Y(1364527)$
$-U_{1342}X(3754261)$	$+U_{1452}[U_{4715}X(3162457)$
$+U_{1453}X(3724561)$	$-2U_{4563}Z(1763254)]$
	$+[Z(1763254)$
	$-3U_{4715}U_{3762}]X(5742361)\}$

Results and Discussion

For comparison of this approximation, which is given in equation (14), and used equation (15) and the result of the six-gluon scattering with exact matrix element squared [6], we used the Monte Carlo program. The calculations are done at the Fermilab Tevatron 1800 Gev c.m. energy. Using the parton density functions of ref. [7] with Λ=0.20 Gev and we applied the following cuts on the outgoing partons

$$P_T^i > 25 \text{ Gev. for each pair of the final jet } \cos\theta_{ij} < \cos\left[\frac{200}{n-2}\right], |\eta_i| < 3.5,$$

where P_T^i denotes the transverse momentum and η_i the rapidity of the outgoing gluon.

The total cross-section rate for the process $gg \rightarrow ggggg$ is listed in Table (4). wjere we omc;ided tje a[[rpxo,atopms wjocj are siggested by Maxwell [5], Kunszt and Stirling [8].

From these comparisons, it is clear that our result has given excellent approximation for the process $gg \rightarrow ggggg$ compared with exact result.

Table 4. The total cross-section rates (in nb) for different approximations compared with exact cross-section rate for the process $gg \to ggggg$.

Method	$gg \to ggggg$
Exact	$0.91 \pm .003$
Makhshoush	$0.90 \pm .003$
Maxwell	0.81 ± 0.03
K-Stirling	1.33 ± 0.04

References

[1] Berends, F.A.; Gele, W.T. and Kuijf, H., "Exact and Approximate Expressions for Multi-gluon Scattering." *Nucl. Phys.*, B 333, (1990), 120.

[2] Parke, S.J. and Taylor, T.R. "Amplitude for N-gluon Scattering." *Phys. Rev. Lett,* 56, (1986), 2459.

[3] Mangano, Parke. S. and Xu., " Duality and Multi-gluon Scattering." *Nucl. Phys.*, B 298, (1988), 653.

[4] Makhshoush, K. "Calculation for Six-gluon Scattering to Leading Order in the Number of Colour." Papers submitted to the Journal of Science, King Saud University., "Accepted".

[5] Maxwell, C.J. "Approximation N-jet Production." *Phys. Lett.*, B192, (1987), 190.

[6] Kleiss, R. and Kuijf, H. "Multi-gluon Cross-section and Five-jet Production at Hedron Colliders." *Nucl. Phys.,* B321 (1984) 616.

[7] Duke, D.W. and Owens, J.F. "FSU-HEP-831115." *Phys. Rev.*, D 30, (1984) 149.

[8] Kunszt, Z. and Stirling, W.J. "Proceddings of the Workshop on Physics at Future Accelerotor." *La Thuile and Geneva,* J. H. Mulvey (Ed.), 2, (July 1987), 548.

تقريب تشتت سبعة قلونات

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ملخص البحث. في هذا البحث أعطينا صيغة مقرّبة لمربّع السعة لسبعة قلونات. يعتمد هذا التقريب على معامل الانخفاض ونتائج تشتت ستة قلونات، ويبيّن هذا التقريب أنّ نسبة الخطأ تنخفض إلى حوالي ١٪ إذا ما قورن بالنتائج الكليّة لتشتت سبعة قلونات.

قارنّا الصيغة التحليليّة التقريبيّة التي حصلنا عليها للمقطع العرضي مع المقطع العرضي الكليّ فوجدنا أنّ هناك توافقًا في حدود ٩٩٪ وبالتالي أمكن استخدام هذه الصيغة التحليليّة في دراسة ٥ ـ جت بدلا من الصيغة الكليّة لطول صيغتها الرياضيّة وإحصائيّة الحوادث فيها محدودة جدًا.

اكتشفنا أيضًا أنّ التقريب المقترح بوساطة سترلنج يحتوي على معامل ثابت. هذا المعامل لا يعتمد على تفاصيل ـ الجت، ومن المُسَلَّم به أنّ مدى صحة أي تقريب يكون حسّاسًا إلى تفاصيل ـ القطع إلى حد ما.

وبالتالي فإنّ التقريب المصنّف من قِبل سترلنج غير صحيح لكل حادثة على حدة في الطور الفضائي.