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On NBUA Class of Life Distributions

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Abstract. The NBUA class of life distributions, which is an intermediate class between NBU and NBUE, and its dual class NWUA are studied. Relationships with other classes of life distributions are presented. Closure of the NBUA class under formation of parallel systems is investigated. A shock model in which shocks are arriving according to a nonhomogeneous Poisson process, is also studied. The Laplace transform characterization for this class is established.

1. Introduction

In many reliability applications, various classes of life distributions and their duals have been introduced to descibe several types of deterioration or improvement that accompany ageing. It has been found very useful to classify life distribution using the concept of stochastic ordering. For definitions of several classes of life distributions e.g. IFR, IFRA, NBU, DCCS, NBUC, NBUAS, NBUFR, NBUE, HNBUE, DMRL and their duals, see [1-7].

Let Y be a non-negative random variable representing a device life with distribution F(t) and survival function $\bar{F}(t) = 1 = F(t)$ and let Y_t be the residual life of the device of age t with distribution $F_t(y)$ and survival function

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$$\bar{F}_t(y) = \begin{cases} \bar{F}(t+y)/\bar{F}(t), & \bar{F}(t) > 0 \\ 0 & \bar{F}(t) = 0. \end{cases} \tag{1.1}$$

It is well known that F belongs to the IFR (DFR) if Y_t is decreasing (increasing) in $t \geq 0$ in stochastic ordering, F belongs to the NBUC (NWUC) class if Y_t is smaller (larger) than Y for any $t \geq 0$ in convex ordering and F belongs to DCCS (ICCS) class if Y_t is decreasing (increasing) in weak stochastic ordering (wst), *i.e.* Y_{t1}, $(>_{wst})$ $(\leq)$ Y_{t2} for any $0 \leq t_1 \leq t_2 \leq \infty$.

Deshpande, *et al.* [5] introduced another set of classes in terms of stochastic dominance. Their motivation was to relate the use of stochastic dominance in applied economics to notions of ageing in reliability theory. One of their interesting classes are the new better than used of second order NBU(2). However, they did not consider closure of these classes under reliability operations, or shock models. Abouammoh and Ahmed [1] studied NBUAS (NBU(2)) closure properties under some reliability operations such as convolutions, mixtures, coherent systems and homogeneous Poisson shock models. In fact NBUA (NBU(2)) class as will be shown later is a middle class between the NBU and NBUE classes. This of course would prove useful in applications since it is less restrictive than NBU and easier to distinguish in practice than NBUE. The NBU notion of ageing compares a new unit with used units of all possible ages. The NBUE concept does the same for all available new units of all possible ages. One might look at the NBUA property as comparing the average performance of corresponding used units, which is more appealing in practice.

The main theme of this paper is to investigate further this class [NBUA (NBU(2)]. In section 2 definitions and relationships are considered. Section 3 established the closure properties under parallel system. Shock models for nonhomogeneous Poisson process are studied in section 4. In section 5 the Laplace transforms characterization for this class is established.

2. Definition and Relationships

The weak stochastic ordering is related to the convex (variability) ordering (see, Stoyan, [8]) and is defined as follows.

Definition 2.1 A non-negative random variable Y with distribution F is said to be new better than used in average ordering (NBUA) if $Y_t \leq_{wst} Y$ for all $t \geq 0$ and $Y \in$ NBUA (or $F \in$ NBUA). Its dual class is new worse than used in average ordering (NWUA) which is defined by $Y_t \geq_{wst} Y$ for all $t \geq 0$. The above inequalities are equivalent to

$$\bar{F}(t) \int_0^x \bar{F}(u)\,du \geq (\leq) \int_0^x \bar{F}(t+u)\,du\ ;\ x > 0. \tag{2.1}$$

Theorem 2.1 If Y ∈ NBUA then Y ∈ NBUE.

Proof. Since Y is NBUA,

$$\bar{F}(t) \int_0^x \bar{F}(u)\,du \geq \int_0^x \bar{F}(t+u)\,du \quad \forall\, x \geqslant 0. \tag{2.2}$$

and by letting x tend to ∞ in (2.2), we have

$$\bar{F}(t) \int_0^\infty du \geq \int_0^\infty \bar{F}(t_u)\,du\ ;\ t > 0. \tag{2.3}$$

But

$$\int_0^\infty \bar{F}(u)\,du = \mu.$$

So (2.3) becomes

$$\mu\,\bar{F}(t) \geq \int_0^\infty \bar{F}(t+u)\,du\ ,$$

which is the NBUE condition □.

Theorem 2.2 If Y ∈ DCCS then Y ∈ NBUA.

Proof. Since Y ∈ DCCS,

$$\bar{F}(t_2) \int_0^x \bar{F}(u+t_1)\,du \geq \bar{F}(t_1) \int_0^x \bar{F}(t_2+u)\,d\ ;\ \forall\, x > 0\ ;\ 0 \leq t_1 \leq t_2 \leq x$$

and the result follows by letting t_1 tend to zero. □

Theorem 2.3 If Y ∈ NBU then Y ∈ NBUA.

Proof. It follows from the definitions. □

Then we have the following implications.

$$\text{IFR} \Rightarrow \text{IFRA} \Rightarrow \text{NBU} \Rightarrow \text{NBUA} \Rightarrow \text{NBUE}.$$

A similar chain of implications hold for the corresponding dual classes.

3. Closure of the NBUA Class Under Parallel System

Since Aboaummoh and El-Neweihi [9] have shown that the NBUE class is closed under parallel system of iid components it may be of interest to ask whether or not the same holds for the NBUA class. The following result gives an affirmative answer to this question.

Theorem 3.1 Let $x_1, x_2 \dots, x_n$ be iid with distribution $F(.)$ and $F \in$ NBUA. Then the random variable $X_{(n)} = \max(X_1, X_2, \dots, X_n)$ has distribution $F_n(.) \in$ NBUA.

Proof. We have $F \in$ NBUA if

$$\int_0^x \frac{\bar{F}(t+u)}{\bar{F}(t)}\, du \le \int_0^x \bar{F}(u)du$$

i.e.

$$\int_t^{x+t} \frac{\bar{F}(u)}{\bar{F}(t)}\, du \le \int_0^t \bar{F}(u)\, du + \int_t^{t+x} \bar{F}(u)\, du$$

i.e.

$$\int_t^{x+t} \left[\frac{1-F(u)}{1-F(t)} - (1-F(u))\right] du \le \int_0^t [1-F(u)]\, du.$$

Since F is a distribution function, we have

$$\int_0^t (1-F(u))du \leq \int_0^t [1-F^n(u)]du. \tag{3.1}$$

and

$$\int_t^{x+t} \frac{[1-F(u)]F(t)du}{[1-F(t)]} \geq \int_t^{x+t} \frac{F^n(t)[1-F^n(u)]du}{[1-F^n(t)]} \tag{3.2}$$

Since

$$\int_t^{x+t} \left[\frac{F(t)[1-F(u)]}{[1-F(t)]} - F^n(t) \frac{[1-F^n(u)]}{[1-F^n(t)]} \right] du \geq 0$$

holds it follows that

$$\int_t^{x+t} \frac{F(t)[1-F(u)]}{1-F(t)} \left\{ 1 - F^{n-1}(t) \frac{(1-F^n(u))}{(1-F(u))} \cdot \frac{(1-F(t))}{(1-F^n(t))} \right\} \Bigg] du$$

$$= \int_t^{x+t} \left[\frac{F(t)(1-F(u))}{(1-F(t))} \left\{ 1 - F^{n-1}(t). \frac{1+F(u)+\cdots+F^{n-1}(u)}{1+F(t)+\cdots+F^{n-1}(t)} \right\} \right] du$$

$$\geq \int_t^{x+t} \left[\frac{F(t)[1-F(u)]}{[1-F(t)]} \left\{ 1 - F^{n-1}(t). \frac{1+F^{-1}(t)+\cdots+F^{n-1}(t)}{1+F(t)+\cdots+F^{(n-1)}(t)} \right\} \right] du \geq 0$$

Since $F(u) \leq F^{-1}(u) \leq F^{-1}(t)$ for $u \leq t$, from (3.1) and (3.2) we have

$$\int_t^{x+t} \frac{F^n(t)[1-F^n(u)]\,du}{[1-F^n(t)]} \leq \int_0^t (1-F^n(u))\,du$$

$$\Longleftrightarrow \int_t^{x+t} \frac{F_n(t)\,\bar{F}_n(u)\,du}{\bar{F}_n(t)} \leq \int_0^t \bar{F}_n(u)\,du.$$

The above inequality may be written as follows:

$$\int_t^x \frac{\bar{F}_n(u)}{\bar{F}_n(t)} \leq \int_0^t \bar{F}_n(u)\,du + \int_t^x \bar{F}_n(u)\,du$$

whence

$$\int_0^x \frac{\bar{F}_n(u+t)\,du}{\bar{F}_n(t)} \leq \int_0^x \bar{F}_n(u)\,du$$

showing that $F_n(\cdot) \in$ NBUA.

4. A Nonhomogeneous Poisson Shock Model

Let $\bar{H}(t)$ be the survival function of a device which is subject to a sequence of independent shocks occurring randomly in time. Let $N = \{N(t), t \geq 0\}$ be a general counting process during [0,t] and $\{\bar{P}_k\}_{k=0}^{\infty}$ the probability that the device survives the first k shocks. $\bar{P}_k$ is assumed to be decreasing in k and $\bar{P}_0 = 1$. The the survival probability that the device survives beyond time t can be expressed in the form

$$\bar{H}(t) = \sum_{k=0}^{\infty} P\{N(t)=k\}\,\bar{P}_k \tag{4.1}$$

For $P\{N(t)=k\} = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$, this shock model was considered by Essary *et al.* [10]. A-Hameed and Proschan [11] and [12] have studied this model for cases when N is a nonhomogeneous Poisson process and N is a birth process respectively. These authors have considered the cases where $\bar{P}_k$, k, = 0, 1, ... has the following discrete properties IFR, IFRA, NBU and DMRL.

In this section we consider the shock model given by (4.1) such that shocks occur according to a non-homogeneous poisson process with mean value function $\Lambda(t)$ and event rate $\lambda(t)$ and event rate $\lambda(t) = \Lambda'(t)$ both defined on $[0, \infty)$: $\lambda\Lambda'(0)$ is taken as right derivative of $\Lambda(t)$ at t=0. Thus the shock model (4.1) is reduced to the form

$$\bar{H}(t) = \sum_{k=0}^{\infty} e^{-\Lambda(t)} \frac{\Lambda^k(t)}{k!} \bar{P}_k \tag{4.2}$$

Now we prove that the discrete NBUA property of = 0,1,2, ... is preserved for $\bar{H}(t)$ under the model (4.2).

Definition 4.1 A discrete distribution or its survival $\bar{P}_k = 1 - P_k$, k = 0,1,2, ... , is called discrete NBUA (NWUA) if

$$\sum_{k=0}^{l} \bar{P}_{j+k} \le (\ge)\ \bar{P}_j \sum_{k=0}^{l} \bar{P}_k \, ; l > 0. \tag{4.3}$$

Theorem 4.1 The survival function $\bar{H}(t)$ in (4.2) is NBUA (NWUA) if $\{\bar{P}_k\}_{k=0}^{\infty}$ has the discrete NBUA (NWUA) property (4.3), $\Lambda'(t) > 0$ for $t \ge 0$ and

$$\bar{H}_l(t) \{ \sum_{j=0}^{\infty} \Gamma x(j+1) / \Lambda'(0) + V_x / \Lambda'(x)\} \le$$

$$\bar{P}_l \{ \sum_{j=0}^{\infty} \bar{P}_j \Gamma x(j+1) / \Lambda'(0) + V_x / \Lambda'(t+x)\} \tag{4.4}$$

its be

$$\bar{H}_l(t) = \sum_{j=0}^{\infty} e^{-\Lambda(t)} \frac{(\Lambda(t))^j}{j!} .\bar{P}_{j+l} \, ; \Gamma x(j+1) = \int_0^x e^{-t} \frac{t^j}{j!} dt$$

$$V_x = \int_{\xi}^{x} e^{-t} \frac{t^j}{j\,i} \, dt; \; 0 \leqslant \xi \leqslant x, l > 0.$$

Proof. Using Definition (4.1) $\{\bar{P}_k\}_{k=0}^{\infty}$ is NBUA gives that

$$\sum_{k=0}^{l} \bar{P}_{j+k} \leq \bar{P}_j \sum_{k=0}^{l} \bar{P}_k$$

then

$$\bar{P}_j \geq (\bar{P}_j + l\bar{P}_{j+l})/(1 + l\bar{P}_1)$$

then

$$\bar{P}_l \, \bar{P}_j \geq \bar{P}_{j+l} \qquad j = 0,1,2, \dots, l > 0 \tag{4.5}$$

multiplying both sides of (4.5) by $e^{-\Lambda(t)} \dfrac{(\Lambda(t))^j}{j!}$ and summing over $j = 0,1,2, \dots$ gives

$$\bar{P}_l \, \bar{H}(t) \geq \bar{H}_l(t) \tag{4.6}$$

Where

$$\bar{H}_l(t) = \sum_{j=0}^{\infty} e^{-\Lambda(t)} \frac{\Lambda^j(t)}{j\,i} \bar{P}_{j+l}$$

In fact $\bar{H}(t)$ is NBUA if

$$\bar{H}(t) \geq \int_0^x \bar{H}(t+u)\,du \Big/ \int_0^x \bar{H}(u)\,du \tag{4.7}$$

Note that (4.7) is satisfied if

$$\frac{\bar{H}_l(t)}{\bar{P}_l} \leq \int_0^x \bar{H}(t+u)\,du \Big/ \int_0^x \bar{H}(u)\,du$$

i.e.

$$\frac{\bar{H}_1(t)}{\bar{P}_1} \leq \int_0^x \sum_{j=0}^{\infty} e^{-\Lambda(t+u)} \frac{(\lambda(t+u))^j}{j!} \frac{d\Lambda(t+u)}{\Lambda'(t+u)} \Big/ \int_0^x \sum_{j=0}^{\infty} e^{-\Lambda(k)} \frac{(\lambda(u))^j}{j!} \frac{d\Lambda(u)}{\Lambda'(u)} \tag{4.8}$$

where

$$\frac{d\Lambda(u)}{du} = \Lambda'(u) \Longleftrightarrow du \frac{d\Lambda(u)}{\Lambda'(u)} .$$

Applying the second mean value Theorem on the right side of (4.8) and using $\Lambda'(0) > 0$ gives (4.4) and the proof is completed.

The survival function $\bar{H}$ (t) in model (4.2) is (NWUA) if $\{ \bar{P}_k \}_{k=0}^{\infty}$ is NWUA, $\Lambda'(0) > 0$, $t \geq 0$ and condition (4.4) is satisfied with the inequality sign reversed.

5. Laplace Transform for NBUA

Here we establish necessary and sufficient conditions for a life distribution to have the NBUA property by using the Laplace transform. These conditions may be used to investigate closure under some reliability operations.

Now let F be a distribution function such that $F(0-) = 0$ and $\phi(s) = \int_0^{\infty} e^{-su} dF(u)$, $s > 0$ be the Laplace transform of F(x). Define

$$a_n(s) = \frac{(-1)^n}{n!} \frac{d^n}{ds^n} \left(\frac{1 - \phi(s)}{s} \right), n \geq 0, s \geq 0 \tag{5.1}$$

let $a_{n+1}(s) = s^{n+1} a_n(s)$ for $n \geq 0$ and $a_0(s) = 1$. The transforms $a_n(s)$ can be written in the forms

$$a_n(s) = \frac{1}{n!} \int_0^{\infty} u^n e^{-su} \bar{F}(u) du \tag{5.2}$$

and

$$a_{n+1}(s) = \frac{1}{n!} \int_0^{\infty} s(su)^n e^{-su} \bar{F}(u) du, \quad n \geq 0, s \geq 0 \tag{5.3}$$

Vinogradov [13] has characterized the IFR property in terms of α_n(s while Block and Savits [14] have obtained similar characterizations for IFRA, DMRL, NBU and NBUE properties. Abouammoh *et al.* [15] have characterized the NBAFR property similarly. Here we establish a similar characterization for the NBUA properties.

Theorem 5.1 Let F be life distribution with F (0 −) = 0, then F has NBUA (NWUA) property if and only if

$$\sum_{j=n+1}^{\infty} a_{j+1}(s) \geq (\leq) (1/\bar{F}(t)) \sum_{j=n+1}^{\infty} a'_{j+1}(s) \tag{5.4}$$

its be

$$a'_{j+1} = \int_0^{\infty} \frac{(sx)^j}{j!} e^{-sx} \bar{F}(t+x)\, dx$$

Proof. Assume for the necessary conditioin that F is NBUA. Then the form (5.3) gives

$$\sum_{j=n+1}^{\infty} a_{j+1}(s) = \sum_{j=n+1}^{\infty} \int_0^{\infty} \frac{(sx)^j}{j!} e^{-sx} \bar{F}(x)\, dsx$$

$$= s \int_0^{\infty} \left(\sum_{j=n+1}^{\infty} \frac{(sx)^j}{j!} e^{-sx} \right) \bar{F}(x)\, dx$$

$$= s \int_0^{\infty} \bar{F}(x) \left(\int_x^{\infty} \frac{(su)^n}{n!} e^{-sx}\, du \right) dx$$

$$= s \int_0^{\infty} \frac{(sx)^n}{n!} e^{-sx} \int_0^{x} \bar{F}(u)\, du dx \tag{5.5}$$

Since F is NBUA we have

$$\sum_{j=n+1}^{\infty} a_{j+1}(s) \geq \frac{s}{\bar{\bar{F}}(t)} \int_0^{\infty} \frac{(sx)^n}{n!} e^{-sx} \int_0^{x} \bar{F}(t+u)\, dudx$$

i.e.

$$\sum_{j=n+1}^{\infty} a_{j+1}(s) \geq \frac{1}{\bar{\bar{F}}(t)} \int_0^{\infty} \bar{F}(t+x) \left(\int_x^{\infty} \frac{(sx)^n}{n!} e^{-su}\, dsu \right) dx$$

$$= \frac{1}{\bar{\bar{F}}(t)} \sum_{j=n+1}^{\infty} \left(\int_0^{\infty} \frac{(Sx)^j}{j!} e^{-sx} \bar{F}(t+x)\, dx \right.$$

This is equivalent to (5.4) which is the necessary condition. Now, we show the sufficiency of the condition (5.4). Equation (5.4) states

$$\int_0^{\infty} \sum_{j=n+1}^{\infty} \frac{(su)^j}{j!} e^{-sx} \bar{F}(u) du \geq \frac{1}{\bar{F}(t)} \sum_{j=n+1}^{\infty} \left(\int_0^{\infty} \frac{(sx)^j}{j!} e^{-sx} \bar{F}(t+x)\, dx \right.$$

or

$$\int_0^{\infty} G_n(u) \bar{F}(u)\, du \geq \frac{1}{\bar{F}(t)} \int_0^{\infty} G_n(u) \bar{F}(t+u)\, du \qquad (5.6)$$

where

$$G_n(u) = \sum_{j=n}^{\infty} \frac{(su)^{j-1} e^{-su}}{(j-1)!},$$

Substituting s = n/t, we have

$$G_n(u) = \sum_{j=n}^{\infty} \frac{(nu/t)^{j-1} e^{-su}}{(j-1)!},$$

which means that G_n (.) is gamma distribution with characteristic function

$$\phi_n(w) = (1 - \frac{iwt}{n})^{-n}.$$

This implies that G_n (.) converges to the degenerate distribution

$$G(x) = \begin{cases} 1 & \text{for } u \leq x \\ 0 & \text{for } u > x \end{cases}$$

The proof of the NWUA case consists of reversing all inequalities in the above proof.

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References

[1] Abouammoh, A.M. and Ahmed, A.N. "Two Notions of Positive Ageing Based on Stochastic Dominance". *Statistische Hefte,* 30 (1989), 214-229.

[2] Ahmed, A.N. "The Generalized HNBUE (HNWUE) Class of Life Distributions". *Methods and Models of Operation Research,* 34 (1990), 183-184.

[3] Barlow, R.E. and Proschan, F. *Statistical Theory of Reliability and Life Testing: Probability Models.* To Begin With, Silver Spring, MD, 1981.

[4] Bryson, M.C. and Siddiqui, M.M. "Some Criteria for Ageing". *J. Amer. Statist. Assoc.* 64 (1969), 1472-1483.

[5] Deshpends, J.M.; Kochar, S.C. and Singh, H. "Aspects of Positive Ageing". *J. Amer. Statist. Assoc.* 64 (1986), 1472-1483.

[6] Hendi, M.I. "On Decreasing Cumulative Conditional Survival Classes of Life Distributions". *Pak. J. Statist.* 8(3)B(1992), 1-8.

[7] Klesfjo, B. "HNBUE Survival Under Some Shock Models". *Scand. J. Statist.,* 8 (1980), 811.814.

[8] Stoyan, D. *Comparison Methods for Queues and Other Stochastic Models.* New York: John Wiley, 1983.

[9] Abouammoh, A.M. and El-Neweihi, E. "Closure of the NBUE and DMRL, Class Under Formation of Parallel System, *Stat. & Prob. Lett.,* 4 (1986), 223-224.

[10] Esary, J.D.; Mershall, A.W. and Proschan, F. "Shock Models and Wear Process" *Ann. Prob.*, 1 (1973), 627-648.

[11] A-Hameed, M.S. and Proschan, F. "Nonstationary Shock Models". *Stock. Proc. Appl.*, 1 (1973), 333.404.

[12] A-Hameed, M.S. and Proschan, F. "Shock Models with Underlying Birth Process" *J. Appl. Prob.*, 12 (1975), 13-28.

[13] Vinogradov, O. "The Definition of Distribution Function with Increasing Hazard Rate in Terms of Laplace Transform". *Theor. Prob. Appl.*, 18 (1973), 811-814.

[14] Block, H. and Savits, T. "Laplace Transforms for Classes of Life Distribution". *Ann. Prob.*, 7 (1980), 911-916.

[15] Abouammoh, A.M.; Hendi, M.I. and Ahmed, A.V. "Stock Models with NBUFR and NBAFR Survivals". *Trabajos de Estadestica,* 3 (1988), 97-113.

دراسة في فصل توزيعات الحياة الجديد أفضل من المستخدم في المتوسط (ج ف ت م)

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(سُلِّمَ في ٣ ربيع ثاني ١٤١٣هـ؛ وقُبل للنشر في ٢٥ ذي الحجة ١٤١٣هـ)

ملخص البحث. يعتبر فصل الحياة الجديد أفضل من المستخدم في المتوسط (ج ف ت م) كحالة وسط بين فصليّ الحياة: الجديد أفضل من المستخدم (ج ف ت) والجديد أفضل من المستخدم في التوقع (ج ف ت ت). يشمل هذا البحث دراسة كلٍ من: علاقة فصل (ج ف ت م) بالفصول الأخرى، خاصية إنغلاق فصل (ج ف ت م) تحت نظام اتّصال على التوازي لوحداته، نموذج صدمات يتعرض لصدمات طبقًا لعملية بواسون غير المتجانسة وتمييز تحويلات لبلاس لفصل (ج ف ت م).