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Representation of a Harmonic Functon with a Singularity of Finite Order

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Abstract. Using some standard methods from the theory of distributions, a harmonic function in $\mathbb{R}^n/\{0\}$ whose singularity at 0 is of finite order is expressed in terms of E_n, the fundamental singularity for the Laplacian, and its derivatives. This result is then generalized to polyharmonic functions.

1. Preliminaries

Let ∂_k denote the differential operator $\partial/\partial x_k$ in $\mathbb{R}^n$, with $1 \le k \le n$ and $n \ge 2$. If $\mu = (\mu_1, \ldots, \mu_n)$ is an n – tuple of non-negative integers, ∂^μ will denote the operator $\partial_1^{\mu_1} \ldots \partial_n^{\mu_n} = \partial^{|\mu|} / \partial^{\mu_1} x_1 \ldots \partial^{\mu_n} x_n$, where $|\mu| = \mu_1 + \ldots + \mu_n$. Hence $\Delta = \Sigma_1^n \partial_k^2$ is the Laplacian operator and $\Delta^k = \Delta\Delta^{k-1}$, $k \ge 2$.

If Ω is an open set in $\mathbb{R}^n$ then, following the standard notation of distributions, $D(\Omega)$ will denote the topological linear space of real C^∞ functions with compact support in Ω. A distribution in Ω is a continuous linear functional on $D(\Omega)$, i.e. an element of the dual space $D'(\Omega)$. By defining

$$\|\phi\|_m = \sup\left\{\left|\partial^\mu \phi(x)\right| : x \in \Omega, |\mu| \le m\right\}, \quad m \ge 0,$$

for any $\phi \in D(\Omega)$, a linear function T on $D(\Omega)$ is a distribution of finite order in Ω if, given a compact set $K \subset \Omega$, there is an integer $m \ge 0$ and a finite constant M such that

$$|T(\phi)| \le M\|\phi\|_m \qquad (1)$$

for all $\phi \in D(\Omega)$ with supp $\phi \subset K$. The *order* of the distribution T is the smallest value of m for which the inequality (1) holds for all K. If T is of order m, it is immediately obvious from the relation $\partial^{\mu}T(\phi) = (-1)^{|\mu|}T(\partial^{\mu}\phi)$ that $\partial^{\mu}T$ is of order $m + |\mu|$.

The fundamental singularity for the Laplacian Δ at 0 is the function defined on $I\!R^n/\{0\}$ by

$$E_n(x) = \begin{cases} \dfrac{1}{2\pi}\log|x| & \text{if } n = 2 \\ -\dfrac{1}{(n-2)\sigma_n |x|^{n-2}} & \text{if } n \geq 3, \end{cases}$$

where σ_n is the area of the unit sphere in $I\!R^n$. It satisfies $\Delta E_n = \delta$ in the sense of distributions, where δ is the Dirac measure supported at 0. The corresponding singularity for Δ^2, denoted by $E_{n,2}$, is also a function of $|x| = r$, and may be constructed by solving the differential equation

$$\Delta E_{n,2}(r) = \frac{1}{r^{n-1}}\left(r^{n-1}E'_{n,2}(r)\right)' = E_n(r)$$

in $r > 0$. Similarly, for $p \geq 2$, $E_{n,p}$ satisfies $\Delta E_{n,p} = E_{n,p-1}$ and hence $\Delta^p E_{n,p} = \delta$.

Harmonic Distributions in $I\!R^n/\{0\}$ of Finite Order in $I\!R^n$

If f is a locally integrable function in $I\!R^n$, then the linear functional $\phi \to \langle f, \phi \rangle$ defined by

$$\langle f, \phi \rangle = \int_{I\!R^n} f(x)\phi(x)dx$$

is clearly a distribution of order 0, for $\langle f, \phi \rangle \leq \|\phi\|_0 \int_k f(x)dx$ where $K = \text{supp } \phi$. But if f is singular at a point, say 0, and locally integrable in $I\!R^n/\{0\}$, then, depending on the strength of the singularity, the order of the distribution defined by f may be equal to or greater than 0. One measure of the order of the singularity of f at $x = 0$ is the smallest value of m which makes $|x|^m f(x)$ bounded in a neighborhood of 0. But since a locally integrable function need not be bounded, a sharper criterion for determining the order of the distribution is the integrability of $|x|^m f(x)$ in a neighborhood of 0.

Lemma 1

Let f be a locally integrable function in $I\!R^n/\{0\}$ which is not integrable in the unit ball $|x| < 1$. If there is a positive integer m such that $|x|^m f(x)$ is integrable in $|x| < 1$, then f extends to a distribution in $I\!R^n$ of order $\leq m$.

Proof

Let $\phi \in D(I\!R^n)$ be arbitrary. Since ϕ has compact support, there is a positive number $a > 1$ such that $\phi(x) = 0$ in $|x| \geq a$. By Taylor's formula,

$$\phi(x) = \sum_{|\mu| \leq m-1} \frac{1}{\mu!} \partial^\mu \phi(0) x^\mu + \sum_{|\mu|=m} \frac{1}{\mu!} \partial^\mu \phi(tx) x^\mu, \quad x \in \mathrm{IR}^n$$

for some $t \in (0,1)$, where $\mu! = \mu_1! \ldots \mu_n!$ and $x^\mu = x_1^{\mu_1} \ldots x_n^{\mu_n}$.

Now we define the linear functional

$$< f, \phi > = \int_{|x|>1} f(x)\phi(x)dx + \int_{|x|\leq 1} f(x)\left[\phi(x) - \sum_{|\mu|\leq m-1} \frac{1}{\mu!} \partial^\mu \phi(0) x^\mu\right]\delta\xi$$

$$= \int_{|x|>1} f(x)\phi(x)dx + \int_{|x|\leq 1} f(x) \sum_{\mu=m} \frac{1}{\mu!} \partial^\mu \phi(tx) x^\mu dx.$$

Since $|x^\mu| = |x_1|^{\mu_1} \ldots |x_n|^{\mu_n} \leq |x|^{|\mu|}$, we have

$$< f, \phi > \leq \|\phi\|_0 \int_{1<|x|<a} |f(x)| dx + c\|\phi\|_m \int_{|x|\leq 1} |x|^m |f(x)| dx, \tag{2}$$

where c is a positive constant that depends on m and n. By hypothesis the two integrals in (2) are finite, hence f defines a distribution in $I\!R^n$ of order $\leq m$.

In lemma 2 it is important to note that m is an upper bound on the order of the distribution f, therefore the best estimate is obtained when m is the smallest positive integer for which $|x|^m f(x)$ is locally integrable in $I\!R^n$. In $I\!R^2$, for example, both $\log|x|$ and $1/|x|$ are locally integrable, hence they define a distribution of order 0, but $1/|x|^2$ is of order 1.

In $I\!R^n$, $n \geq 3$, $E_n(|x|)$ is a constant multiple of $|x|^{-n+2}$, hence the integral

$$\int_{|x|\leq 1} |x|^{-m} E_n(|x|) dx = \sigma_n \int_0^1 r^{-m} E_n(r) r^{n-1} dr$$

converges if and only if $-m - n + 2 + n - 1 = -m + 1$ is greater than -1, *i.e.* if and only if $m < 2$. Consequently both $E_n(r)$ and $r^{-1}E_n(r)$ represent distributions of order 0 in $I\!R^n$, $n \geq 2$, while $r^{-m}E_n(r)$ with $m \geq 2$ is of order $m-1$.

Now since $\partial^\mu E_n$ is a function of order $r^{2-n-|\mu|}$ near $r = 0$, where $n \geq 2$ and $|\mu| \geq 1$, we have

Lemma 2

The function $\partial^{\mu} E_n$ for $|\mu| \geq 1$ extends to a distribution in $\mathbb{R}^n$ of order $|\mu| - 1$.

Note that the notion of "order" as it applies to the singularity of the function $\partial^{\mu} E_n$ at 0 is not the same as the order of the distribution $\partial^{\mu} E_n$. Now we can state our main result:

Theorem 1

let u be a harmonic function in $\mathbb{R}^n / \{0\}$ such that $|x|^m u(x)$ is integrable in $|x| < 1$, $m \geq 0$. Then u is represented in $\mathbb{R}^n / \{0\}$ by the sum

$$u(x) = \sum_{|\mu| \leq m+1} c_{\mu} \partial^{\mu} E_n + h, \tag{3}$$

where c_{μ} are constants and h is a harmonic function in $\mathbb{R}^n$

Proof

Since a harmonic function in $\mathbb{R}^n / \{0\}$ is locally integrable in $\mathbb{R}^n / \{0\}$, lemma 1 implies that u extends to a distribution in $\mathbb{R}^n$, also denoted by u, of order $\leq m$. Consequently the distribution Δu, being 0 in $\mathbb{R}^n / \{0\}$, has its support in $\{0\}$, and is therefore a finite linear combination of δ and its derivatives (see [1], for example). But since Δu is of order $\leq m+2$, it can only have the form

$$\Delta u = \sum_{|\mu| \leq m+2} c_{\mu} \partial^{\mu} \delta, \tag{4}$$

where c_{μ} are constants. The general solution of the differential equation (4) is given by

$$u = E_n * \sum_{|\mu| \leq m+2} c_{\mu} \partial^{\mu} \delta + h, \tag{5}$$

where h is a harmonic function in $\mathbb{R}^n$ and * denotes the convolution operation.

Using the well known properties of convolution products (see[1]), the right-hand side of (5) can be expressed as

$$\begin{aligned} u &= \sum_{|\mu| \leq m+2} c_{\mu} \partial^{\mu} E_n * \delta + h \\ &= \sum_{|\mu| \leq m+2} c_{\mu} \partial^{\mu} E_n + h \end{aligned}$$

But since u is of order $\leq m$, we conclude from lemma 2 that $c_\mu = 0$ for all $|\mu| = m+2$.

In $I\!R^n$, the representation (3) can be shown to lead to the finite sum

$$u(r,\theta) = h(r,\theta) + a_0 \log r + \frac{1}{r}(a_1 \cos\theta + b_1 \sin\theta) + \ldots$$
$$+ \frac{1}{r^{m+1}}(a_{m+1} \cos(m+1)\theta + b_{m+1} \sin(m+1)\theta)$$

which coincides with the well-known Laurent series type expansion of u about $x = 0$ when $|x|^m u(x)$ is stipulated to be integrable in $|x| < 1$. Thus the sum over μ in the right hand side of (3) represents the singular part of u in that expansion. The significance of theorem 1 is that we have arrived at this representation by relying only on some standard results from the theory of distributions. In particular, neither analytic function theory nor potential theory was used to prove (3).

In general, if u is a harmonic function in $I\!R^n / \{0\}$ which extends to a distribution in $I\!R^n$, then the distribution Δu is supported in $\{0\}$. Consequently Δu, and hence u, is a *finite* linear combination of derivatives of E_n. But this implies that the singularity of u at 0 is of finite order. Thus, in general, a harmonic function in $I\!R^n / \{0\}$ does not extend to a distribution in $I\!R^n$. The extension is possible if and only if the singularity at 0 is of finite order.

Further Generalization

If u satisfies $\Delta^p u = 0$ in $I\!R^n / \{0\}$ for some positive integer p and $|x|^m u(x)$ is integrable in $|x| < 1$, then

$$\Delta^p u = \sum_{|\mu| \leq m+2p-1} c_\mu \partial^\mu \delta,$$

and therefore

$$u = \sum_{|\mu| \leq m+2p-1} c_\mu \partial^\mu E_{n,p} + h_p,$$

where h_p is a polyharmonic function in $I\!R^n$ which satisfies $\Delta^p h_p = 0$.

In theorem 1 we can obviously replace $I\!R^n$ by any open set $\Omega \subset I\!R^n$, 0 by any point $x_0 \in \Omega$, and $|x| < 1$ by any neighborhood of x_0 in Ω. Thus we obtain

Corollary

Let u be a function in $\Omega \setminus \{x_0\}$ where it satisfies $\Delta^p u = 0$ for some positive integer p. If $|x - x_0|^m u(x)$ is integrable in a neighborhood of x_0, then there are constants c_μ and a polyharmonic function h_p in Ω, where $\Delta^p h_p = 0$, such that

$$u(x) = \sum_{|\mu| \leq m+2p-1} c_\mu \partial^\mu E_{n,p}(|x - x_0|) + h_p(x) \qquad in \ \Omega \setminus \{x_0\}$$

Here, again, a comparison with the series expansion of u in [2] shows that we have arrived at the same representation under the integrability condition imposed on u.

References

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[2]Aronszajn, N.; Creese, T.M. and Lipkin, L.J. *Polyharmonic Functions* . Oxford: Clarendon Press, 1983.

تمثيل الدالّة التوافقية حول نقطة شاذة ذات رتبة منتهية

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ملخص البحث. يعالج هذا البحث تمثيل الدالّة التوافقية حول نقطة الأصل عندما تكون نقطة الأصل شاذة برتبة منتهية. وفي سبيل ذلك يستخدم أساليب نظرية التوزيعات المتعارف عليها، ويظهر التمثيل المطلوب بدلالة الحل الأساسي لمعادلة لابلاس ومشتقّاته. ثم يتطرق البحث إلى تعميم هذه النتيجة للدوال ذات التوافق المضاعف.