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Mathematical Model for Flow of Pseudoplastic Fluids in Porous Media

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Abstract. A non-linear partial differential equation for the flow of all types of pseudoplastic fluids in porous media was derived. The rheological characteristics of the pseudoplastic fluids were described by the generalized approach developed by Metzner and Reed. A linear form of the equation developed was derived and solved. Analytical solutions were obtained for a constant injection rate into an infinite reservoir.

The validity of the present analysis was verified by comparing the pressures calculated from the developed equation with published data. The equation deduced provides an accurate prediction for pressure distributions.

Nomenclature

C = fluid compressibility, Pa^{-1}

C_t = system compressibility, Pa^{-1}

D_p = particle diameter, m

h = formation thickness, m

K = Consistency coefficient, $Pa.S^n$

k_r = permeability in the radial (horizontal) direction, m^2

K' = Metzner and Reed parameter, $Pa.s^{n'}$

n' = Metzner and Reed parameter

P = pressure, Pa

P_D = non-Newtonian dimensionless pressure

P_i = initial pressure, Pa

P_{wf} = flowing bottomhole pressure, Pa
$\triangle P$ = Pressure difference, $p_{wf} - p_i$, Pa
q = flow rate, m^3/s
r = radial distance, m
r_D = dimensionless radial distance
r_w = wellbore radius, m
t = time, s
t_D = non-Newtonian dimensionless time
u = superficial velocity (volumetric rate of flow per unit cross-sectional area), m/s
u_r = superficial velocity in the radial (horizontal) direction, m/s
V = pore velocity, m/s
$\dot{\gamma}$ = shear rate, s^{-1}
$\Gamma(x)$ = Gamma function.
μ_{eff} = effective viscosity (a viscosity level parameter for pseudoplastic fluids), $Pa.s^{n'}.m^{1-n'}$
λ = group defined by Eq. 12.
ϕ = porosity, fraction
τ = shear stress, Pa

Subscripts

D = dimensionless
r = radial
w = wellbore

Introduction

Non-Newtonian fluids such as polymer solutions, microemulsions and macroemulsions are injected into reservoirs to control mobility and enhance oil recovery. These fluids necessitate a basic understanding of their flow behavior through porous media [1;2]. Several studies of the rheology of non-Newtonian fluids in porous media have been reported in literature [3-17]. Most of these studies have mathematically described the flow of pseudoplastic fluids by either power-law or Ellis model [3-14]. Pascal [15;16], and Al-Fariss and Pinder [17] investigated the flow of yield-fluids through porous media. Thus, most of the equations developed by previous investigators for pseudoplastic fluids were limited to two models: power-law and Ellis models. These models are two of the twenty known rheological models of pseudoplastic fluids [18]. Thus, the equations developed by previous investigators cannot be used to predict the pressures distribution in reservoirs for all types of pseudoplastic fluids.

Therefore, the present analysis was carried out to develop a general equation for all pseudoplastic fluids. The rheological characteristics of those fluids can be described by the generalized flow shear equation developed by Metzner and Reed [19].

$$\tau_w = K' \, (8V/D)^{n'} \tag{1}$$

Rheological Model for Pseudoplastic Fluids

The porous media can be regarded as a conduit with a complicated cross section with mean hydraulic radius R_h equal to R/2 [10]. The superficial velocity (u) can be related to the average pore velocity (V) by the following relation

$$u = \phi \, V \tag{2}$$

The wall shear stress in a porous medium is given by Bird, et al. [20]

$$\tau_w = \frac{2}{25} \, \frac{D_p \, \phi}{(1 - \phi)} \, \frac{\Delta P}{L} \tag{3}$$

The permeability of the porous medium can be defined from Blake-Kozeny Equation [20].

$$K_r = \frac{D_p^2 \; \phi^3}{150\,(1 - \phi)^2} \tag{4}$$

The term 150 is an average value accepted by many investigators since its first use by Ergun [12]. Substituting Eqs. 2,3 and 4 in Eq. 1, the superficial velocity is expressed as

$$u = \left(\frac{K_r}{\mu_{eff}} \, \frac{\Delta P}{L} \right)^{1/n'} \tag{5}$$

where μ_{eff} is given by

$$\mu_{eff} = 12^{n'-1} K' (150 \phi K_r)^{(1-n')/2} \tag{6}$$

For power-law fluids, the values of n' and K' are equal to n and $K [(1+3n)/4n]^n$ respectively. Hence Eq. 6 becomes

$$\mu_{eff} = \frac{K}{12} \left(\frac{3+9n}{n} \right)^n (150 K_r \phi)^{(1-n)/2} \tag{7}$$

Eq. 7 is the same equation obtained by previous investigators [2,11,13] for power-law fluids. In radial coordinates, Eq. 5, can be written as

$$u_r^{n'} = \left(\frac{K_r}{\mu_{eff}} \frac{\partial P}{\partial r} \right) \tag{8}$$

Eq. 8 can be combined with the continuity equation and an equation of state to derive a partial differential equation for flow of all types of pseudoplastic fluids in porous media.

Theoretical Analysis

A partial differential equation that models the flow of pseudoplastic fluids through a porous medium will be derive under the following conditions.

1- The porous medium has a uniform thickness and is homogeneous.

2- Permeability is constant.

3- The compressibility of the fluid and pressure gradient are small.

4- Gravity effect is negligible.

5- The flow of the fluid is isothermal.

The continuity equation for radial flow in a porous medium is given by Ikoku [13] and Desouky [18].

$$\frac{1}{r} \frac{\partial}{\partial r} (r \rho u_r) = -\frac{\partial}{\partial t} (\phi \rho) \tag{9}$$

The equation of state may be written as

$$\rho = \rho_0 e^{C(P-P_0)} \tag{10}$$

Substituting Eqs. 8 and 10 in Eq. 9, and applying the conditions of items (1) through (5), the following equation is obtained.

$$\frac{\partial^2 P}{\partial r^2} + \frac{n'}{r}\,\frac{\partial P}{\partial r} = \lambda\,\frac{\partial P}{\partial t}\,r^{1-n'} \tag{11}$$

where

$$\lambda = 12^{n'-1}\,n'\,\phi\,C_t\,K'\,(q/2\pi h)^{n'-1}\,(150\,\phi\,K_r)^{\frac{1-n'}{2}}\;/\,K_r \tag{12}$$

$$C_t = C + \frac{1}{\phi}\,\frac{\partial\phi}{\partial\rho} \tag{13}$$

For power-low fluids, where $n' = n$ and $K'\,[(1+3n)/4n]^n$, Eq. 11 reduces to the flow equation developed by Ikoku [2].

Dimensional Analysis

It might be advisable to express Eq. 11 in dimensionless form. The following definitions will be used.

1) $P_D = (P-P_i)\,/\,\lambda_1$ (Dimensionless pressure)

where

$$\lambda_1 = \lambda\,q\,r_w^{1-n'}\,/\,(2\pi h\,C_t\,\phi\,n')$$

2) $r_D = r/r_w$ (Dimensionless radius)

3) $t_D = \dfrac{t}{\lambda\,r_w^{3-n'}}$ (Dimensionless time)

4) $$\frac{\partial P}{\partial r} = \frac{\partial P_D}{\partial r_D}\cdot\frac{\partial r_D}{\partial r}\,\lambda_1$$

5) $$\frac{\partial^2 P}{\partial r^2} = \frac{\partial^2 P_D}{\partial r_D^2}\cdot\frac{\partial r_D^2}{\partial r^2}\,\lambda_1$$

6- $$\frac{\partial r_D}{\partial r} = \frac{1}{r_w}$$

7- $$\frac{\partial P}{\partial t} = \frac{\partial P_D}{\partial t_D} \cdot \frac{\partial t_D}{\partial t} \lambda_1$$

8- $$\frac{\partial t_D}{\partial t} = \frac{1}{\lambda r_w^{3-n'}}$$

Substituting the equations defined by items 1 through 8 in Eq. 11, the following linear partial differential equation is obtained.

$$\frac{\partial^2 P_D}{\partial r_D^2} + \frac{n'}{r_D} \frac{\partial P_D}{\partial r_D} = r_D^{1-n'} \frac{\partial P_D}{\partial t_D} \tag{14}$$

Analytical Solution

Equation 14 can be solved for the case of constant injection rate of pseudoplastic fluids into an infinitely large reservoir. The initial and boundary conditions are:

(a) $P_D (r_D , 0) = 0.0$

(b) $(\partial P_D / \partial r_D)_{r_D} = - 1.0,$ for $t_D > 0$

(c) $P_D (r_D , t_D) \rightarrow 0$, as $r_D \rightarrow \propto$

(d) at the wellbore, $r_D = 1$

Using the Laplace transform method [22], applying the conditioins of items a through d, and deriving the inversion of the resulting equation for large times, the following approximate analytical inversion is obtained.

$$P_D = \frac{3-n'^{\frac{2(1-n')}{(3-n)}} \; t_D^{\frac{1-n'}{3-n'}}}{(1-n') \, \Gamma (\frac{2}{3-n'})} \tag{15}$$

In dimensional variables, Eq. 15 becomes

$$P_{wf} = P_i + \left(\frac{q}{2\pi h}\right)^{n'} \frac{12^{n'-1}\ K'}{K_r} \frac{(150\,\phi\,K_r)^{(1-n')/2}}{(1-n')}$$

$$[\,(3-n')^2\, t/\lambda\,]^{(1-n')/(3-n')} \ / \ \Gamma\ \left(\frac{2}{3-n'}\right) \tag{16}$$

Equation 16 represents the flow of any pseudoplastic fluid in a porous medium under the conditions of constant injection rate and an infinitely large reservoir. A graph of log ΔP versus log t yields a straight line of slope $(1-n')/(3-n')$, which can be solved for the value of n'. The intercept at t = 1, may be solved for the value of K' if ϕ, K_r and C_t are known. In case of knowing the rheological properties of the injected fluid, the transient flow behavior of that fluid can be determined.

Determining the Values of n' and K'

Values of n' and K' can be estimated from the measured data of shear stress τ and shear rate $\dot{\gamma}$ using the following Rabinowitsch and Mooney equation (15).

$$\frac{8V}{D} = \frac{4}{\tau_w^3} \int_0^{\tau_w} \tau^2 \dot{\gamma}\, d\tau \tag{17a}$$

where n' and K' are defined as:

$$n' = \frac{ln\,[\,d(\tau_w)\,]}{ln\,[\,d\,(8V/D)\,]} \tag{18}$$

$$K' = \tau_w \,/\, (8V/D)^{n'} \tag{19}$$

In the present analysis, the procedure followed to integrate Eq. 17a is the modified Gauss four-points method suggested by Desouky and El-Emam [23] where Eq. 17a is written as:

$$\frac{8V}{D} = 4\,I\,/\,\tau_w^2 \tag{17b}$$

where I is given by

$$I = 0.173\,[(\tau^2\dot{\gamma})_1 + (\tau^2\dot{\gamma})_4] + 0.326\,[(\tau^2\dot{\gamma})_2 + (\tau^2\dot{\gamma})_3]$$

The procedure for determining 8V/D as a function of shear stress and shear rate is outlined as follows:

1) Square the values of shear stress (τ) and multiply them by the shear rate ($\dot{\gamma}$).
2) Make a suitable correlation relating log ($\tau^2\dot{\gamma}$) and log (τ).
3) At the first experimental data point of τ_1, determine the values of the following four points of ($\tau^2\dot{\gamma}$) from the equations.

   $(\tau^2\dot{\gamma})_1$ is estimated at $\tau = 0.06945\tau_1$

   $(\tau^2\dot{\gamma})_2$ is estimated at $\tau = 0.33\tau_1$

   $(\tau^2\dot{\gamma})_3$ is estimated at $\tau = 0.67\tau_1$

   $(\tau^2\dot{\gamma})_4$ is estimated at $\tau = 0.93055\tau_1$

4) Assume the calculated τ_1 equal to $(\tau_w)_1$, then calculate the first value of (8V/D) from Eq. 17b. The calculated values of (8V/D), and $(\tau_w)_1$ are corresponding to τ_1 and $\dot{\gamma}_1$.
5) Repeat steps (3) and (4) for the remaining experimented data points of τ and $\dot{\gamma}$ to obtain the corresponding converted values of 8V/D and τ_w.

Comparison with the Published Data

To verify the validity of the present analysis, a comparison was made between the pressures calculated from Eq. 16 and the simulated data (2, 24). A biopolymer injection test was performed at a test site during various periods of the injection history. The measured pressures are plotted against time in Fig. 1. The average porosity, thickness and flow rate were found to be 0.15, 4.877 m, and 1.84×10^{-4} m^3/s respectively. The rheological properties of the injected polymer are plotted in Fig. 2. Equations 17 through 19 were used to calculate the values of n′ and K′. These values are 0.323 for n′ and 0.021 Pa.s$^{n'}$ for K′. Equation 16 shows that the slope of a graph (log $\triangle$P − log t) is given by

$$\text{Slope} = \frac{1 - n'}{3 - n'} \tag{20}$$

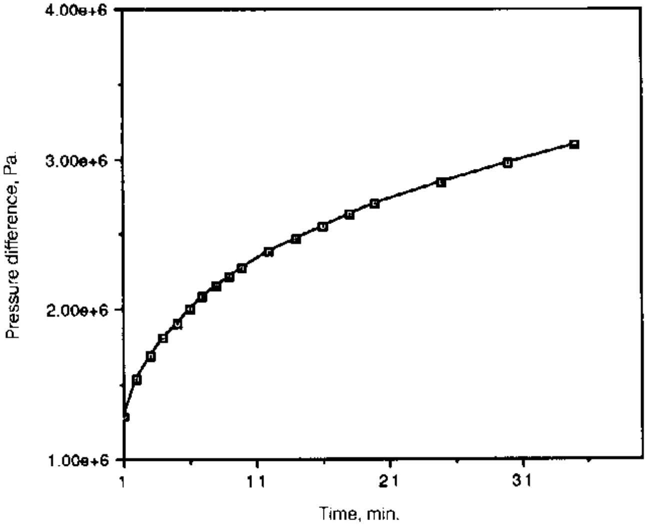

Fig. 1. Pressure difference vs. time for a polymer injection test.

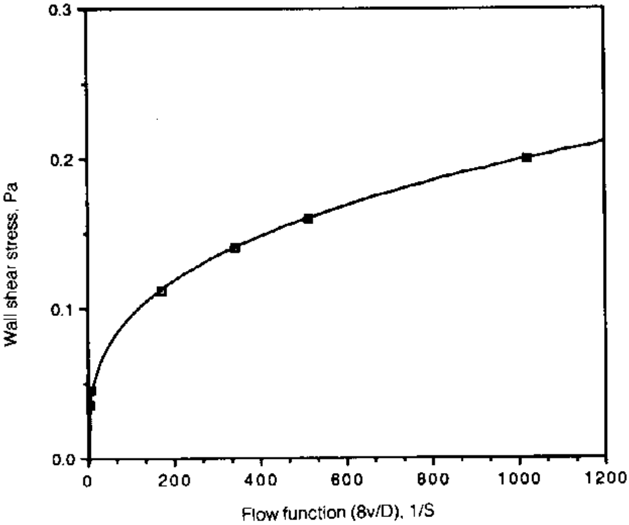

Fig. 2. Rheological data and resulting flow curve.

The value of the slope can be obtained by plotting the measured pressures versus time on log-log scale as shown in Fig. 3. This figure shows that the value of the slope is equal to 0.25. The value of n′ was calculated from Eq. 20 and was found to be 0.333. This indicates that the value of n′ determined from Fig. 3 is consistent with that obtained from rheological data given in Fig. 2. The pressures calculated from Eq. 16 are plotted against the measured ones in Fig. 4. This figure shows that an excellent agreement exists between the pressures calculated from Eq. 16 and the measured ones with an average relative error of 0.32%.

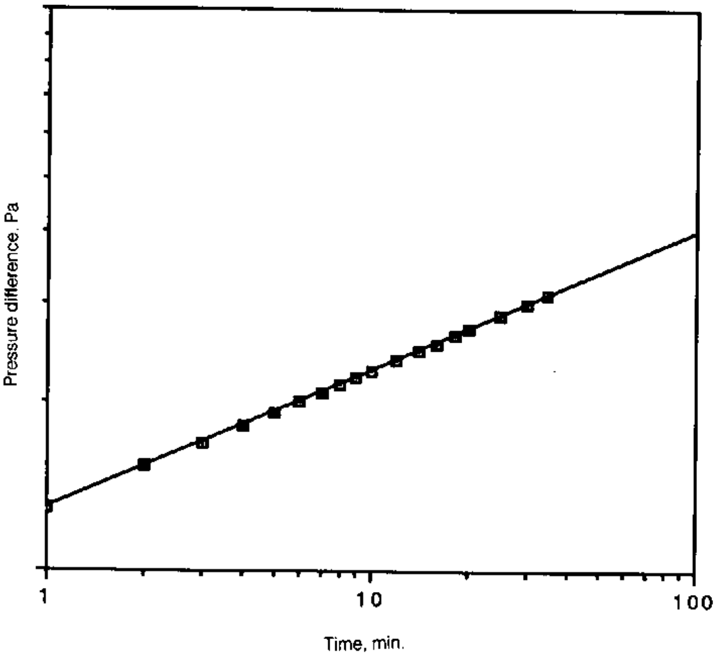

Fig. 3. Logarithm of pressure difference vs. logarithm of time for a polymer injection test.

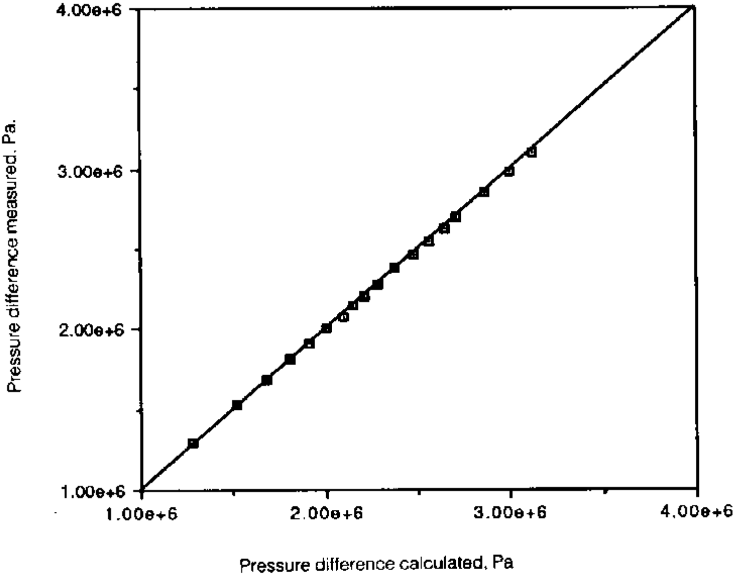

Fig. 4. Cross plot for the measured and calculated pressure drop.

Conclusion

Flow of pseudoplastic fluids in porous media is expressed by a simple mathematical formula, Eq. 16. It can be used for any type of pseudoplastic fluids irrespective of the rheological model which described the relationship between shear stress and shear rate. The equation developed ables to predict the pressure drop in a porous medium due to flow of a pseudoplastic fluid with an average relative error of 0.32%.

References

[1] Gogarty, W.B. "Mobility Control with Polymer Solutions." *Society of Petroleum Engineers Journal.* 7, No. 2 (June, 1967), 161-173.

[2] Ikoku, C.U. and Ramey, H.J. "Transient Flow of Non-Newtonian Power-law Fluids in Porous Media." *Society of Petroleum Engineers Journal,* 19, No. 3 (June, 1979), 164-173.

[3] Sadowski, T.J. "Non-Newtonian Flow Through Porous Media, II. Experimental." *Transactions of Society of Rheology,* 9, No. 2 (1965), 251-271.

[4] Christopher, R.H. and Middleman, S. "Power-law Flow Through a Packed Tube." *Industrial and Engineering Chemistry Fundamental,* 4, No. 4 (1965), 422-426.

[5] McKinley, R.M.; Jahns, H.O. and Harris, W.W. "Non-Newtonian Flow in Porous Media." *American Institute of Chemical Engineering Journal,* 12, No. 1 (1966), 17-20.

[6] Gaitonde, N.Y. and Middleman, S. "Flow of Viscoelastic Fluids Through Porous Media." *Industrial and Engineering Chemistry Fundamental,* 6, No. 1 (1967), 145-147.

[7] Gogarty, W.B. "Rheological Properties of Pseudoplastic Fluids in Porous Media." *Society of Petroleum Engineers Journal,* 7, No. 1 (June, 1967), 145-160.

[8] Dauben, D.L. and Menzie, D.E. "Flow of Polymer Solutions Through Porous Media." *Journal of Petroleum Technology,* 19, No. 8 (Aug. 1967), 1065-1073.

[9] Van Pollen, H.K. and Jargon, J.R. "Steady-state and Unsteady-state Flow of Non-Newtonian Fluids Through Porous Media." *Society of Petroleum Engineers Journal,* 9, No. 1 (March, 1969), 80-88.

[10] Savins, J.G. "Non-Newtonian Flow Through Porous Media." *Industrial and Engineering Chemistry,* 16, No. 10 (1969), 18-47.

[11] Bondor, P.L.; Hirasaki, G.J. and Tham, M.J. "Mathematical Simulation of Polymer Flooding in Complex Reservoir." *Society of Petroleum Engineers Journal,* 12, No. 5 (1972), 369-382.

[12] Alvarado, D.A. and Marsden, S.S. "Flow of Oil-in-Water Emulsion Through Tubes and Porous Media." Paper SPE 5859 presented at the SPE-AIME *46th Annual California Regional Meeting,* Long Beach, April 8-9, 1976.

[13] Ikoku, C.U. "Transient Flow of Non-Newtonian Fluids in Porous Media." *Ph.D. Dissertation,* Stanford University, Stanford, CA, 1978.

[14] Govier, G.W. and Aziz, K. *The Flow of Complex Mixtures in Pipes.* New York: Van Nostrand Reinhood Co., 1982.

[15] Pascal, H. "Rheological Behavior Effect of Non-Newtonian Fluids on Steady and Unsteady Flow Through a Porous Medium." *Int. J. for Numerical and Analytical Methods in Geomechanics,* 7, No. 1 (1983), 289-303.

[16] Pascal, H. "Nonsteady Flow of Non-Newtonian Fluids Through a Porous Medium." *Int. J. Eng. Sci.,* 21, No. 3 (1983), 199-210.

[17] Al-Fariss, T. and Pinder, K.L. "Flow Through Porous Media of a Shear-Thinning Liquid with Yield Stress." *The Canadian Journal of Chemical Engineering*, 65, No. 6 (June, 1987), 391-405.

[18] Desouky, S.E.M. "A New Laminar – Turbulent – Transition Criterion for Pseudoplastic Fluids." *J. of Pet. Sc. & Eng.*, 5, No. 3 (1991), 285-291.

[19] Metzner, A.B. and Reed, J.C. "Flow of Non-Newtonian Fluids-Correlation of the Laminar, Transition and Turbulent Flow Regions." *American Institute of Chemical Engineers Journal,* 14, No. 5 (1955), 434-440.

[20] Bird, R.B.; Stewart, W.E. and Lightfoot, E.N. *Transport Phenomena.* New York: John Wiley & Sons, Inc., 1960.

[21] Ergun, S. "Fluid Flow Through Packed Columns." *Chemical Eng. Prog.*, 48, No. 3 (1952), 89-94.

[22] Van Everdingen, A.F. and Hurst, W. "The Application of Laplace Transformation to Flow Problems in Reservoirs." *Transactions of AIME,* 207 (1949), 305-324.

[23] Desouky, S.E.M. and El-Emam, N.A. "Program Designs for Pseudoplastic Fluids." *Oil and Gas Journal,* 87, No. 51 (Dec. 1989), 48-51.

[24] Odeh, A.S. and Yang, H.T. "Flow of Non-Newtonian Power-law Fluids Through Porous Media." *Society of Petroleum Engineers Journal,* 19, No. 3 (June, 1979), 155-163.

نموذج رياضي لسريان السوائل اللدائنيّة الكاذبة في الأوساط المساميّة

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(استُلم في ١٩٩٣/٢/١٤م؛ قُبل للنشر في ١٩٩٣/٧/٣م)

ملخص البحث. في هذا البحث استنبطت معادلة تفاضليّة جزئيّة غير خطيّة لسريان السوائل اللانيوتونيّة (اللدائنيّة الكاذبة) في الأوساط المسامية. عُبِّر عن الخواص الريولوجية لهذه السوائل بالطريقة العامة لمزنر وريد. واقترحت حلول تحليلية خطيّة للمعادلة المستنبطة عند معدّل حقن ثابت داخل المكمن اللاحدودي.

تم اختبار دقة المعادلة المقترحة وذلك بمقارنة النتائج المحسوبة منها بالبيانات المنشورة.