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New Recursive Formulas for Eigenvalue Sensitivity Analysis

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Abstract. Eigenvalue sensitivity analysis represents a key discipline in many engineering system applications where the dynamic behavior of such systems is closely related to the eigenvalues of the system (state) matrix. Currently available eigenvalue sensitivity evaluation methods are based on a general form of the system matrix derivatives with respect to the sensitivity parameter of interest. In many engineering systems, however, the structure of the system matrix is such that its derivatives with respect to practical system parameters constitute special rank-one forms. This paper presents a novel application of a compact matrix exchange formula to the eigenvalue sensitivity problem, which includes rank-one derivative matrices, leading to very fast recursive sensitivity formulas with substantial savings in computation time and memory requirements. With such recursive formulas, the evaluation of higher-order eigenvalue sensitivities is therefore attainable using previously calculated lower-order sensitivities. An illustrative application to power system dynamic stability analysis is presented in which the new eigenvalue sensitivity formulas were successfully used to estimate the effect of parameter changes on the dynamic system modes.

Introduction

Practical engineering systems are often subjected to variations in the system and operating conditions and are, therefore, characterized by an inherent dynamic behavior in response to such variations [1]. The eigenvalues of the so-called system (or state) matrix represent key indicators of such dynamic behavior. Eigenvalue calculations are currently involved in many engineering applications of modern control theory including system design, control strategies and modelling [2].

The dynamic behavior of an engineering system would obviously depend on the specific values and settings of various design and operating parameters of the system. Consequently, the elements of the system matrix are, in general, functions of such design and operating parameters. During the system design phase, the engineer usually performs many dynamic performance analyses associated with combinations of different values of the design parameters before arriving at the final system design. On the other hand, most engineering systems experience variations in the operating and environmental parameters which occur continuously during their in-service operation. Such variations may reflect changes made by the system operator or random fluctuations in the environment in which the system is operating.

Regardless of the source of parameter variations, proper assessment of the system dynamic performance should take into account all variations (or combinations of variations) of the different system design and operating parameters which are likely to occur in practice. In the conventional approaches, such an assessment usually involves a large number of eigenvalue analyses of the system matrix computed at different assumed parameter values. This task consumes significant amount of computer resources, especially for large-scale systems.

Alternatively, the eigenvalue sensitivity analysis can be used to assess the system dynamic performance for many parameter variations with the eigenvalues (dynamic modes) of the system matrix computed only once at the nominal parameter values while first, second and higher order changes in the eigenvalues (associated with the successive terms of Taylor's series expansion) are computed using much faster sensitivity formulas. Of course, in order to maintain a reasonable accuracy of the results, higher order sensitivity calculations would be required for larger parameter deviations from their nominal values.

Many enhancements have been made to eigenvalue sensitivity methodologies in both theory and applications [3-5]. In many of today's engineering system applications, eigenvalue sensitivity analysis is regarded as an important discipline in the design and operation studies. Because many practical engineering systems today are large-scale in nature, efficient computation of the eigenvalue sensitivities with respect to various operating and design parameters represents a key requirement in the analysis. In this respect, it is believed that much faster computations and less memory requirements can be attained if the special characteristics and structure pertaining to the system under study are utilized.

The work of this paper is based on the observation that, in many practical engineering systems, only very few elements of the system matrix would depend on a given parameter that is likely to change in practice. In other words, the first and higher-order derivatives of the system matrix with respect to the parameter would be extremely sparse with only a few non-zero elements. Therefore, such derivative matrices are likely to constitute special rank-one forms [6]. Under this condition, and while still yielding exact (non-approximate solutions), the eigenvalue sensitivity formulas can be reduced to compact, very fast and elegant expressions which can be easily coded in the computer programs. Moreover, the resulting expressions for second-order and higher-order eigenvalue sensitivities constitute very fast recursive schemes involving mostly scalar operations using previously computed lower-order sensitivities.

The novel eigenvalue sensitivity algorithms presented in this paper are believed to have useful applications in a wide range of engineering systems which exhibit rank-one structures of the system matrix derivatives. In addition to avoiding repeated full-scale eigenvalue evaluations associated with many different parameter values, they offer much faster eigenvalue sensitivity computations as compared with other existing methods. When only few eigenvalues of the system matrix are to be examined (corresponding, for example, to the most dominant dynamic modes), the new eigenvalue sensitivity algorithms could offer an additional advantage as they constitute sensitivity formulas for individual eigenvalues.

The paper presents the theoretical basis and implementation of the new analytical formulas for eigenvalue sensitivity computations which lead to substantial savings in the computation time and memory requirements, especially for large-scale systems.

Conventional Eigenvalue Sensitivity Formulas

We consider an (nxn) system matrix A with eigenvalues λ_i; $i = 1,...,n$ arranged in a column vector λ. The eigenvalues, which are assumed to be distinct are related to the corresponding eigenvectors u_i; $i = 1,...,n$ of A by the equations

$$A\,u_i = \lambda_i\,u_i;\ i = 1,...,n \tag{1}$$

Similarly, for the transpose of A, which has eigenvalues ν_j, we can write

$$A^T\,\nu_j = \lambda_j\,\nu_j,\ j = 1,...,n \tag{2}$$

We note that u_i and ν_j are orthogonal and can be scaled such that

$$\nu_i^T u_i = 1 \tag{3-a}$$

and

$$\nu_j^T u_i = 0 \quad \text{for } j \neq i \tag{3-b}$$

Now, differentiating Eq. (1) with respect to a parameter ξ of interest, we get

$$\dot{A}u_i + A\dot{u}_i = \dot{\lambda}_i u_i + \lambda_i \dot{u}_i \tag{4}$$

where

$$\dot{A} = (\partial A / \partial \zeta), \quad \dot{u}_i = (\partial u_i / \partial \zeta)$$

and

$$\dot{\lambda}_i = (\partial \lambda_i / \partial \zeta)$$

Pre-multiplying, Eq. (4) by ν_i^T, we get

$$v_i^T \dot{A} u_i + v_i^T A \dot{u}_i = \dot{\lambda}_i v_i^T u_i + \lambda_i v_i^T \dot{u}_i$$

or

$$v_i^T \dot{A} u_i + (v_i^T A \dot{u}_i - \lambda_i v_i^T \dot{u}_i) = \dot{\lambda}_i v_i^T u_i$$

or

$$v_i^T \dot{A} u_i + (v_i^T A - \lambda_i v_i^T) \dot{u}_i = \dot{\lambda}_i (v_i^T u_i)$$

But, from the transpose of Eq. (2), we get

$$v_i^T A = \lambda_i v_i^T \quad \text{for all j}$$

Hence,

$$v_i^T \dot{A} u_i = \dot{\lambda}_i (v_i^T u_i)$$

and using Eq. (3-a), we get

$$\dot{\lambda}_i = v_i^T \dot{A} u_i \tag{5}$$

Differentiating Eq. (5) again, using the same notation,

$$\ddot{\lambda}_i = \dot{v}_i^T \dot{A} u_i + v_i^T \ddot{A} u_i + v_i^T \dot{A} \dot{u}_i \tag{6}$$

Since u_j ; j = 1,...,n are assumed independent, then $\dot{u}_i$ can be expressed in terms of u_j ; j = 1,..., n as

$$\dot{u}_i = \sum_j \alpha_{ij} u_j \tag{7}$$

where the coefficients α_{ij} can be obtained by substituting Eq. (7) in Eq. (4) and using Eqs. (1) and (3),

$$\alpha_{ij} = (v_j^T \dot{A} u_i) / (\lambda_i - \lambda_j); \;\; j \neq i \tag{8}$$

We note that while α_{ii} is not defined, it will not be needed in subsequent analysis. Using similar argument, v_i can be expressed in terms of v_j ; j = 1,..., n as

$$\dot{v}_i = \sum_j \gamma_{ij} v_j \tag{9}$$

where

$$\gamma_{ij} = -\alpha_{ji} = v_i^T \dot{A} u_j / (\lambda_i - \lambda_j); \;\; j \neq i \tag{10}$$

Therefore, from Eq. (6)

$$\ddot{\lambda}_i = v_i^T \ddot{A} u_i + 2 \sum_{j \neq i} \alpha_{ij} \alpha_{ji} (\lambda_j - \lambda_j) \tag{11}$$

Equations (5) and (11) give the first and second eigenvalue sensitivities as currently known in the literature [3-6] with, sometimes, different ways of representation. Also, from the manipulations shown in the Appendix, the third-order sensitivities are given by:

$$\dddot{\lambda}_i = v_i^T\dddot{A}u_i + 3\sum_{j\neq i}[\alpha_{ij}(v_i^T\ddot{A}u_j) - \alpha_{ji}(v_j^T\ddot{A}u_i)] + 6\sum_{j\neq i}\alpha_{ij}\alpha_{ji}(\dot{\lambda}_i - \dot{\lambda}_j)$$
$$+2\sum_{j\neq i}\sum_{\substack{k\neq j\\ k\neq i}}(\alpha_{ij}\alpha_{jk}\alpha_{ki} - \alpha_{ji}\alpha_{ik}\alpha_{kj}).(2\lambda_k - \lambda_i - \lambda_j) \quad (12)$$

Rank-One Matrix Exchange Formula

We consider the case where the system matrix derivatives $\dot{A}$ and $\ddot{A}$ are (n x n) rank-one matrices of the forms

$$\dot{A} = zx^T \text{ and } \ddot{A} = zy^T \quad (13)$$

where z, x and y are n-column vectors and T denotes transposition. Then

$$(h^T\dot{A}g)(q^T\dot{A}p) = (q^T\dot{A}g)(h^T\dot{A}p) \quad (14\text{-a})$$

$$(h^T\dot{A}g)(q^T\ddot{A}p) = (q^T\dot{A}g)(h^T\ddot{A}p) \quad (14\text{-b})$$

for any n-column vectors g, h, p and q.

The proof is readily seen by direct substitution of Eq. (13) into Eq. (14) and comparing the individual scalar dot-product of vector pairs in the right and left hand sides.

We now show that the application of the rank-one matrix exchange formula (14) has a powerful impact on the eigenvalue sensitivity formulas derived in the previous section. First consider the term $\alpha_{ij}\ \alpha_{ji}$ in the second-order sensitivity expression of Eq. (11), using Eq. (8)

$$\alpha_{ij}\alpha_{ji} = [(v_j^T\dot{A}u_i)(v_i^T\dot{A}u_j)]/[(\lambda_i - \lambda_j)(\lambda_j - \lambda_i)]$$

By letting $g = u_i$, $h = v_j$, $p = u_j$ and $q = v_i$ we get from Eq. (14-a)

$$\alpha_{ij}\alpha_{ji} = [(v_i^T\dot{A}u_i)(v_j^T\dot{A}u_j]/[(\lambda_i - \lambda_j)(\lambda_j - \lambda_i)]$$

and from Eq. (5), written for both i and j and multiplied, we get

$$\alpha_{ij}\alpha_{ji} = \dot{\lambda}_i\dot{\lambda}_j/[(\lambda_i - \lambda_j)(\lambda_j - \lambda_i)] \quad (15)$$

Hence, for a rank-one $\dot{A}$ matrix, Eq. (11) reduces to

$$\ddot{\lambda}_i = v_i^T \ddot{A} u_i + 2\sum_{j\neq i} \dot{\lambda}_i \dot{\lambda}_j / (\lambda_i - \lambda_j) \tag{16}$$

That is, the second-order sensitivities can be obtained directly using first-order sensitivities already calculated.

We shall now show how the application of Eq. (14) is extended to the third and higher-order sensitivities. For the third-order sensitivity of Eq. (12), we first consider the term

$$\alpha_{ij}\alpha_{jk}\alpha_{ki} = [(v_j^T \dot{A} u_i)(v_k^T \dot{A} u_j)(v_i^T \dot{A} u_k)] / [(\lambda_i - \lambda_j)(\lambda_j - \lambda_k)(\lambda_k - \lambda_i)]$$

which is obtained by the direct use of Eq. (8). Applying Eq. (14-a) twice, we get

$$\alpha_{ij}\alpha_{jk}\alpha_{ki} = \dot{\lambda}_i \dot{\lambda}_j \dot{\lambda}_k / [(\lambda_i - \lambda_j)(\lambda_j - \lambda_k)(\lambda_k - \lambda_i)] \tag{17}$$

Similarly, it can be shown that

$$\alpha_{ji}\alpha_{ik}\alpha_{kj} = -\alpha_{ij}\alpha_{jk}\alpha_{ki} \tag{18}$$

Now, the term $\alpha_{ij}(v_i^T \ddot{A} u_j)$ of Eq. (12) can be reduced further if $\dot{A}$ and $\ddot{A}$ have the special relationship as in Eq. (13). In this case,

$$\alpha_{ij}(v_i^T \ddot{A} u_j) = (v_j^T \dot{A} u_i)(v_i^T \ddot{A} u_j) / (\lambda_i - \lambda_j)$$

$$= (v_i^T \dot{A} u_i)(v_j^T \ddot{A} u_j) / (\lambda_i - \lambda_j)$$

which can be written in terms of Eq. (5) as

$$\alpha_{ij}(v_i^T \ddot{A} u_j) = \dot{\lambda}_i (v_j^T \ddot{A} u_j) / (\lambda_i - \lambda_j)$$

Now, we re-write Eq. (16) for the jth eigenvalue as

$$\ddot{\lambda}_j = v_j^T \ddot{A} u_j + 2\sum_{k\neq j} \dot{\lambda}_j \dot{\lambda}_k / (\lambda_j - \lambda_k)$$

or

$$v_j^T \ddot{A} u_j = \ddot{\lambda}_j - 2\sum_{k\neq j} \dot{\lambda}_j \dot{\lambda}_k / (\lambda_j - \lambda_k)$$

hence

$$\alpha_{ij}(v_i^T \ddot{A} u_j) = \frac{\dot{\lambda}_i}{(\lambda_i - \lambda_j)} \left[\ddot{\lambda}_j - 2\sum_{k\neq i} \frac{\dot{\lambda}_j \dot{\lambda}_k}{(\lambda_j - \lambda_k)} \right] \tag{19}$$

Similarly it can be shown that

$$\alpha_{ji}(v_j^T \ddot{A} u_i) = \frac{\dot{\lambda}_j}{(\lambda_j - \lambda_i)}\left[\ddot{\lambda}_i - 2\sum_{k \neq i}\frac{\dot{\lambda}_i \dot{\lambda}_k}{(\lambda_i - \lambda_k)}\right] \tag{20}$$

Using Eq. (15) and Eqs. (17)-(20), the third-order expression Eq. (12) reduces to

$$\dddot{\lambda}_i = v_i^T \dddot{A} u_i + 3\sum_{j \neq i}\left[\frac{(\dot{\lambda}_i \ddot{\lambda}_j + \dot{\lambda}_j \ddot{\lambda}_i)}{(\lambda_i - \lambda_j)}\right] \tag{21}$$

$$-2\sum_{j \neq i}\sum_{\substack{k \neq i \\ k \neq j}}\left[\frac{\dot{\lambda}_i \dot{\lambda}_j \dot{\lambda}_k (2\lambda_k - \lambda_i - \lambda_j)}{(\lambda_i - \lambda_j)(\lambda_j - \lambda_k)(\lambda_k - \lambda_i)}\right]$$

which, again is recursive and depends on the previously computed first and second-order sensitivities.

We note here that the general sensitivity expressions (11) and (12) require the calculation of coefficients α_{ij}, where the main computational effort is expended, while the sensitivity expressions (16) and (21) do not require the coefficients α_{ij} and are therefore much faster to compute.

Applications

As was described in the previous sections, the eigenvalue sensitivity analysis involves the computation of the eigenvalues of the system matrix once at the nominal parameter values and the subsequent application of the eigenvalue sensitivity formulas to evaluate first, second and higher order changes in the eigenvalues subject to assumed variations in the system design and operating parameters. The use of higher order sensitivities is required to achieve more accurate estimates of eigenvalue changes for a given parameter deviation from its nominal value. Conversely, for a given level of accuracy, higher order sensitivities would allow larger deviations in system parameters to be considered.

The authors of this paper have analyzed many engineering systems of different sizes in order to assess the saving in computer resources associated with the use of rank-one matrix formulas for eigenvalue sensitivities as compared to conventional formulas. The size of the system matrix in their studies ranged from 5 to 100; and only the cases which exhibited rank-one derivative matrices were considered. While the use of the rank-one matrix formula for first-order sensitivity calculation does not provide any savings, its use for the second and third-order sensitivities offers remarkable savings in the computational time.

Even when a very efficient computational scheme was employed (for example, by computing various matrix-vector products once and storing them for multi-use in computations), the conventional formulas took about 2.5 times as much as rank-one formulas (in terms of CPU time for both second and third order sensitivities) on a computer work-station for (5x5) system matrices. The ratio was 6.0 for (20x20) system matrices, 13.5 for (50x50) matrices and 26.5 for (100x100) matrices. Indeed, such savings would represent a significant improvement, especially when many design and operational parameters are to be considered in the sensitivity analysis.

Power system application

We now consider an engineering power system modelled as a generator connected to an infinite bus as shown in the figure below. This simple system is widely used in power system studies related to system stability analysis. The power generator is equipped with an exciter to regulate the output voltage. The dynamic behaviour of this system is governed by eleven differential equations, five of which belong to the generating (synchronous) machine itself (including two damper windings), two describe the mechanical torque and the remaining four describe the excitation system.

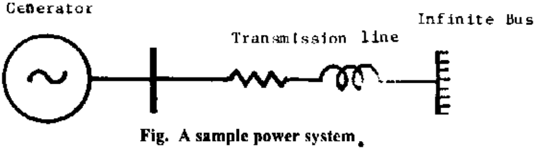

Fig. A sample power system.

Therefore, the system matrix of this problem would be an (11x11) real matrix with eleven real and/or complex (complex conjugate pairs) eigenvalues (system modes). In the majority of power systems, the eigenvalues are distinct. Furthermore, the system matrix A can be evaluated as a function of two matrices P and Q as follows

$$A = P^{-1} Q$$

The elements of the matrices P and Q are, in general functions of the operation and design parameters of the system. The first-order derivative of A with respect to parameter ζ is then given by the solution of the sets of linear equations

$$P\dot{A} = -\dot{P}A + \dot{Q}$$

where $\dot{P} = \partial P / \partial \zeta$, $\dot{Q} = \partial Q / \partial \zeta$ and $\dot{A} = \partial A / \partial \zeta$. The derivative matrices $\dot{P}$ and $\dot{Q}$ as well as P and Q are all evaluated at the nominal parameter values. Using the standard formulation of [7, p. 287-292], both $\dot{P}$ and $\dot{Q}$ are square (11 x 11) matrices. The eleven system differential variables are, respectively, the direct-axis current i_d, field current i_F, direct-axis damper winding current i_D, quadrature-axis current i_q, quadrature-axis damper winding current i_Q, machine angular speed ω machine angular rotor position δ and four excitation-related differential variables V_1, V_3, V_R and E_{FD}. For the sample power system under consideration, the matrices P and Q evaluated at nominal parameter values are given by

$$
P = \begin{bmatrix}
-2.100 & -1.550 & -1.550 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
12.501 & -4.9565 & 77.116 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
22.755 & 4.3694 & -96.255 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2.040 & -1.490 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1.490 & -1.526 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1783.9 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
-0.035 & 0 & 00.051 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}
$$

and

$$
Q = \begin{bmatrix}
0.021 & 0 & 0 & 2.040 & 1.490 & 1.430 & -1.024 & 0 & 0 & 0 & 0 \\
0 & 0.00074 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.0008 \\
0 & 0 & 0.013 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2.100 & -1.550 & -1.550 & 0.021 & 0 & -1.039 & -1.397 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.054 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.014 & -0.362 & -0.362 & -1.428 & -0.790 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0.052 & 0 & 0 & 0.032 & 0 & -0.056 & 0.088 & 0.265 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.004 & -7.9\text{x}10^{-7} & -2.0\text{x}10^{-7} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.122 & 2.122 & 0.053 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.005 & 1.4\text{x}10^{-3}
\end{bmatrix}
$$

Therefore, the system matrix A evaluated at nominal parameter values is given by

$$
A = \begin{bmatrix}
-36.068 & 0.4397 & 1.413 & -3487.9 & -2547.6 & -2445.2 & 1751.4 & 0 & 0 & 0 & -04914 \\
12.501 & -4.9565 & 77.116 & 1208.9 & 882.98 & 847.47 & -0.6070 & 0 & 0 & 0 & 5.5487 \\
22.755 & 4.3694 & -96.255 & 2200.5 & 1607.3 & 1542.6 & -1104.9 & 0 & 0 & 0 & -4.8829 \\
3588.3 & 2648.5 & 2648.5 & -36.050 & 90.038 & 1775.8 & 2386.6 & 0 & 0 & 0 & 0 \\
-3503.6 & -2585.9 & -2585.9 & 35.198 & -123.27 & -1733.8 & -2330.2 & 0 & 0 & 0 & 0 \\
-0.0078 & -0.2030 & -0.2030 & -0.8006 & -0.4430 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1.00 & 0 & 0 & 0 & 0 & 0 \\
130.30 & 133.81 & 133.33 & 87.877 & 92.789 & 230.52 & -27.791 & -265.26 & 0 & 0 & 0.0235 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3.7099 & 0.00079 & -0.00024 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2122.1 & -2122.1 & -53.052 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 53.052 & -1.3761
\end{bmatrix} \text{x } 10^{-3}
$$

The eigenvalues for this matrix (computed at nominal parameter values) are shown in Table 1. They include five real modes and three pairs of complex-conjugate modes.

The authors would like to note here that the majority of system parameters in this application (including, for example, all excitation-related parameters) would lead to a rank-one system matrix derivative Λ and, therefore, the fast eigenvalue sensitivity formulas presented in this paper can be used. As an illustration, the detailed results for one parameter will be shown. We shall consider variations in the regulator stabilizing circuit time constant τ_F. Table 2 shows the nominal value of the τ_F parameter as well as the elements of the p and Q matrices which are affected by variations in the parameter. Table 3 contains first, second and third-eigenvalue sensitivities with respect to the parameter τ_F while Table 4 shows the percentage error obtained when using such sensitivities in Taylor series to obtain estimated new eigenvalues for large-changes of 10% and 40% in the parameter. Only modes which are mostly affected by parameter variations are considered for error analysis. These most sensitive modes are often referred to as the dominant modes which influence the dynamic performance of the system. They are usually the focus of the system

Table 1. Nominal Eigenvalues of 11-mode test power system

λ_1 = -0.0359+j0.9983	λ_4 = - 0.1217	λ_7 = -0.0015-j0.0289	λ_{10} = -0.0004-j0.0067
λ_2 = -0.0359-j0.9983	λ_5 = - 0.0986	λ_8 = -0.0548	λ_{11} = -0.0037
λ_3 = -0.2653	λ_6 = - 0.0015+j0.0289	λ_9 = -0.0004+j0.0067	

Table 2. Sensitivity data for parameter τ_F (nominal value = 0.715)

Element	Expression	Nominal Value
$q_{9,9}$	$0.00265/\tau_F$	0.0037
$q_{9,10}$	$0.565\times10^{-6}/\tau_F$	0.79×10^{-6}
$q_{9,11}$	$0.143\times10^{-6}/\tau_F$	0.2×10^{-6}

Table 3. Sensitivity results for parameter τ_F

Mode	Eigenvalue Sensitivities		
#	1st Order	2nd Order	3rd Order
1	$-8.555\times10^{-16}+j6.726\times10^{-16}$	$2.4\times10^{-15}-j1.873\times10^{-15}$	$-1.01\times10^{-14}+j7.82\times10^{-15}$
2	$-8.555\times10^{-16}-j6.726\times10^{-16}$	$2.4\times10^{-15}+j1.873\times10^{-15}$	$-1.01\times10^{-14}-j7.82\times10^{-15}$
3	1.2×10^{-10}	-3.402×10^{-10}	1.447×10^{-9}
4	-6.856×10^{-10}	1.98×10^{-9}	-8.557×10^{-9}
5	2.924×10^{-7}	-8.495×10^{-7}	3.702×10^{-6}
6	$2.719\times10^{-8}-j4.226\times10^{-8}$	$-9.052\times10^{-8}+j1.073\times10^{-7}$	$4.341\times10^{-7}-j3.974\times10^{-7}$
7	$2.719\times10^{-8}+j4.226\times10^{-8}$	$-9.052\times10^{-8}-j1.073\times10^{-7}$	$4.341\times10^{-7}+j3.974\times10^{-7}$
8	-4.892×10^{-5}	1.468×10^{-4}	-6.615×10^{-4}
9	$1.106\times10^{-5}+j1.328\times10^{-5}$	$-7.483\times10^{-6}-j4.287\times10^{-5}$	$-5.635\times10^{-5}+j1.545\times10^{-4}$
10	$1.106\times10^{-5}-j1.328\times10^{-5}$	$-7.483\times10^{-6}+j4.287\times10^{-5}$	$-5.635\times10^{-5}-j1.545\times10^{-4}$
11	5.215×10^{-3}	-1.464×10^{-2}	6.167×10^{-2}

design and operating engineers in their analyses.

From the results of Table 4, it is clear that the use of eigenvalue sensitivities to estimate changes in system modes for large-parameter changes would, for the given parameter, lead to accurate results and, therefore, would avoid repeat eigenvalue calculations. The size of the estimation error would depend on the size of the parameter change and the order of sensitivity used. The results of Table 4 show that, for a 40% change in τ_F, the percentage error drops from 16.1% to 2.6% when estimating the change in the real mode 11 (the 11th eigenvalue) as the sensitivity order used increases from first to third. From a practical point of view, this represents a significant improvement in accuracy as a result of using higher order sensitivities as it enables the system engineer to estimate, within only 2.6% error (97.4% accuracy), the changes in this dominant dynamic mode for variations of up to 40% in the parameter τ_F.

Table 4. Percentage large-scale estimation error (w.r.t. exact change) for parameter τ_F.

Mode #	+ 10% Change						+ 40% Change					
	1st Order		2nd Order		3rd Order		1st Order		2nd Order		3rd Order	
	Real	Imag	Real	Imag	Real	Imag	Real	Imag	Real	Imag	Real	Imag
9	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.02	0.025	0.007	0.015	.002
10	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.02	0.025	0.007	0.015	.002
11	1.0		0.1		0.01		16.1		6.47		2.6	

In this application, the CPU-time required to calculate second-order and third-order eigenvalue sensitivities for each parameter of interest using rank-one formulas was only 12% (for second-order) and 30% (for third-order) of that using conventional formulas.

Conclusions

The new rank-one matrix exchange formula presented in this paper is believed to have useful applications in a wide class of eigenvalue-related engineering problems. As was demonstrated in the paper, the application of the rank-one formula can lead to a more efficient evaluation of eigenvalue sensitivities, especially for large system matrices. The resulting expressions for second-order and higher-order eigenvalue sensitivities reduce to very fast recursive schemes involving mostly scalar operations using previously computed lower-order sensitivities.

With the use of higher-order eigenvalue sensitivities, more accurate estimates of eigenvalue changes can be obtained for variations in the system design and operating parameters. Although the application of the formula requires a special rank-one structure of the system matrix derivatives, the scope of the formula is believed to cover a wide range of engineering systems in practice including, for example, the dynamic stability analysis of power systems. Analysis of several engineering systems of different sizes has shown that the rank-one sensitivity formulas provide savings in excess of 90% for large-scale applications involving system matrices of sizes greater than (50x50). These savings are realized for each sensitivity parameter of interest leading to a remarkable reduction in the overall computational time.

A wide class of engineering and design problems can be analyzed using the new recursive sensitivity formulas. In the specific application to power system dynamic analysis presented in the paper, very good estimates of eigenvalue changes can be obtained for changes in system parameters of up to ± 25% without the need to repeat the entire eigenvalue calculations. In addition, the use of third-order sensitivities would, in most cases, constitute major improvement in accuracy as compared to first or second-order computations.

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Appendix
Derivation of Third-Order Sensitivities

In order to derive third-order eigenvalue sensitivities, the second-order sensitivities of Eq. (11) are differentiated yielding

$$\dddot{\lambda}_i = 2\sum_{j\neq i}\left[\dot{\alpha}_{ij}\alpha_{ji}\left(\lambda_j-\lambda_i\right)+\alpha_{ij}\dot{\alpha}_{ji}\left(\lambda_j-\lambda_i\right)+\alpha_{ij}\alpha_{ji}\left(\dot{\lambda}_j-\dot{\lambda}_i\right)\right] \tag{A-1}$$

$$\sum_{j\neq i}\left[\gamma_{ij}\, v_j^T\ddot{A}u_i+\alpha_{ij}\, v_i^T\ddot{A}u_j\right]+v_i^T\dddot{A}u_i$$

Now,

$$\alpha_{ij}\left(\lambda_i-\lambda_j\right)= v_j^T\dot{A}u_i$$

Hence,

$$\dot{\alpha}_{ij}\left(\lambda_i-\lambda_j\right)=-\alpha_{ij}\left(\dot{\lambda}_i-\dot{\lambda}_j\right)+\sum_k\left[\gamma_{jk}\, v_k^T\dot{A}u_i+\alpha_{ik}\, v_j^T\dot{A}u_k\right]+v_j^T\ddot{A}u_i$$

$$=-\left(\dot{\lambda}_i-\dot{\lambda}_j\right)\alpha_{ij}+\sum_k\left[-\alpha_{kj}\, v_k^T\dot{A}u_i+\alpha_{ik}\, v_j^T\dot{A}u_k\right]+v_j^T\ddot{A}u_i$$

$$=-\left(\dot{\lambda}_i-\dot{\lambda}_j\right)\alpha_{ij}+v_j^T\ddot{A}u_i+\sum_{\substack{k\neq i\\ k\neq j}}\left[-\alpha_{kj}\alpha_{ik}\left(\lambda_i-\lambda_k\right)+\alpha_{ik}\alpha_{kj}\left(\lambda_k-\lambda_j\right)\right]$$

$$+\left[-\alpha_{ij}\dot{\lambda}_i+\alpha_{ii}\alpha_{ij}\left(\lambda_i-\lambda_j\right)\right]+\left[-\alpha_{jj}\alpha_{ij}\left(\lambda_i-\lambda_j\right)+\alpha_{ij}\dot{\lambda}_j\right]$$

$$=-2\left(\dot{\lambda}_i-\dot{\lambda}_j\right)\alpha_{ij}+v_j^T\ddot{A}u_i+\sum_{\substack{k\neq i\\ k\neq j}}\left[-\alpha_{kj}\alpha_{ik}\left(\lambda_i-\lambda_k\right)+\alpha_{ik}\alpha_{kj}\left(\lambda_k-\lambda_j\right)\right]$$

$$+\alpha_{ij}\left(\lambda_i-\lambda_j\right).\left[\alpha_{ii}-\alpha_{jj}\right]$$

Similarly,

$$\left(\lambda_j-\lambda_i\right)\dot{\alpha}_{ji}=-2\left(\dot{\lambda}_j-\dot{\lambda}_i\right)\alpha_{ji}+v_i^T\ddot{A}u_j$$

$$+\sum_{\substack{k\neq j\\ k\neq i}}\left[-\alpha_{ki}\alpha_{jk}\left(\lambda_j-\lambda_k\right)+\alpha_{jk}\alpha_{ki}\left(\lambda_k-\lambda_i\right)\right]$$

$$+\alpha_{ji}\left(\lambda_j-\lambda_i\right).\left[\alpha_{jj}-\alpha_{ii}\right]$$

Hence, Eq. (A-1) can be written as

$$\begin{aligned}
\dddot{\lambda}_i &= \sum_j \gamma_{ij} v_j^T \ddot{A} u_i + \sum_j \alpha_{ij} v_i^T \ddot{A} u_j + v_i^T \dddot{A} u_i \\
&+ 2\sum_{j\neq i} -\alpha_{ji}\,\dot{\alpha}_{ij}\left(\lambda_i - \lambda j\right) + 2\sum_{j\neq i} \alpha_{ij}\,\dot{\alpha}_{ji}(\lambda_j - \lambda_i) \\
&\quad + 2\sum_{j\neq i} \alpha_{ij}\alpha_{ji}(\dot{\lambda}_j - \dot{\lambda}_i) \\
&= \sum_j \left[-\alpha_{ji} v_j^T \ddot{A} u_i + \alpha_{ij} v_i^T \ddot{A} u_j\right] + v_i^T \dddot{A} u_i \\
&- 2\sum_{j\neq i} \alpha_{ji} \Big\{ -2(\dot{\lambda}_i - \dot{\lambda}_j)\alpha_{ij} + v_j^T \ddot{A} u_i \\
&+ \sum_{\substack{k\neq i \\ k\neq j}} \left[-\alpha_{kj}\alpha_{ik}\left(\lambda_i - \lambda_k\right) + \alpha_{ik}\alpha_{kj}\left(\lambda_k - \lambda_j\right)\right] \\
&\qquad + \alpha_{ij}\left(\lambda_i - \lambda_j\right)\left(\alpha_{ii} - \alpha_{jj}\right) \Big\} \\
&+ 2\sum_{j\neq i} \alpha_{ij} \Big\{ -2\left(\dot{\lambda}_j - \dot{\lambda}_i\right)\alpha_{ji} + v_i^T \ddot{A} u_j \\
&+ \sum_{\substack{k\neq j \\ k\neq i}} \left[-\alpha_{ki}\alpha_{jk}\left(\lambda_j - \lambda_k\right) + \alpha_{jk}\alpha_{ki}\left(\lambda_k - \lambda_i\right)\right] \\
&\qquad + \alpha_{ji}\left(\lambda_j - \lambda_i\right)\left(\alpha_{jj} - \alpha_{ii}\right) \Big\} \\
&\qquad + 2\sum_{j\neq i} \alpha_{ij}\alpha_{ji}\left(\dot{\lambda}_j - \dot{\lambda}_i\right)
\end{aligned}$$

Rearranging terms, we get

$$\begin{aligned}
\dddot{\lambda}_i &= v_i^T \dddot{A} u_i + \sum_{j\neq i} \left[\alpha_{ij} v_i^T \ddot{A} u_j - \alpha_{ji} v_j^T \ddot{A} u_i\right] \\
&+ \left[\alpha_{ii} v_i^T \ddot{A} u_i - \alpha_{ii} v_i^T \ddot{A} u_i\right] \\
&+ 4\sum_{j\neq i} \alpha_{ij}\alpha_{ji}\left(\dot{\lambda}_i - \dot{\lambda}_j\right) - 2\sum_{j\neq i} \alpha_{ji} v_j^T \ddot{A} u_i \\
&- 2\sum_{j\neq i} \sum_{\substack{k\neq i \\ k\neq j}} \left[-\alpha_{ji}\alpha_{kj}\alpha_{ik}\left(\lambda_i - \lambda_k\right) + \alpha_{ji}\alpha_{ik}\alpha_{kj}\left(\lambda_k - \lambda_j\right)\right] \\
&- 2\sum_{j\neq i} \left[\alpha_{ji}\alpha_{ij}\left(\lambda_i - \lambda_j\right)\left(\alpha_{ii} - \alpha_{jj}\right)\right] \\
&+ 4\sum_{j\neq i} \alpha_{ij}\alpha_{ji}\left(\dot{\lambda}_i - \dot{\lambda}_j\right) + 2\sum_{j\neq i} \alpha_{ij} v_i^T \ddot{A} u_j \\
&+ \sum_{j\neq i} \sum_{\substack{k\neq j \\ k\neq i}} \left[-\alpha_{ij}\alpha_{kj}\alpha_{jk}\left(\lambda_j - \lambda_k\right) + 2\alpha_{ij}\alpha_{jk}\alpha_{ki}\left(\lambda_k - \lambda_i\right)\right]
\end{aligned}$$

$$+2\sum_{j\neq i}\alpha_{ij}\alpha_{ji}(\lambda_j-\lambda_i)(\alpha_{jj}-\alpha_{ii})$$
$$+2\sum_{j\neq i}\alpha_{ij}\alpha_{ji}(\dot{\lambda}_j-\dot{\lambda}_i)$$

Hence,

$$\dddot{\lambda}_i = v_i^T\dddot{A}u_i + 3\sum_{j\neq i}\left[\alpha_{ij}\left(v_i^T\ddot{A}u_j\right)-\alpha_{ji}\left(v_j^T\ddot{A}u_i\right)\right]+6\sum_{j\neq i}\alpha_{ij}\alpha_{ji}\left(\dot{\lambda}_i-\dot{\lambda}_j\right)$$
$$+2\sum_{j\neq i}\sum_{\substack{k\neq j\\ k\neq i}}\left(\alpha_{ij}\alpha_{jk}\alpha_{ki}-\alpha_{ji}\alpha_{ik}\alpha_{kj}\right)\cdot\left(2\lambda_k-\lambda_i-\lambda_j\right)$$

صيغ تكراريّة جديدة لتحليل حساسيّة القيمة المميّزة

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ملخص البحث . يمثّل تحليل حساسية القيمة المميّزة نظامًا رئيسيًا في كثير من تطبيقات النظم الهندسيّة حيث إنّ السلوك الدينامي لتلك الأنظمة يرتبط عن كثب بالقيم المميّزة لمصفوفة النظام (الحالة) . والطرق الحاليّة المتواجدة لتقويم حساسية القيمة المميّزة مبنيّة على شكل عام لمشتقّات مصفوفة النظام بالنسبة لحساسية القيمة المتغيّرة ذات الاهتمام . ومع ذلك ففي كثير من الأنظمة الهندسيّة يتم بناء مصفوفة النظام بحيث إنّ مشتقاتها بالنسبة للقيم المتغيّرة لنظام عملي تؤلّف أشكالاً خاصة ذات رتبة واحدة . وهذه الورقة تقدّم تطبيقًا حديثًا لصيغة مصفوفة تبادليّة متضامة لمعضلة حساسية القيم المميّزة مع مصفوفات مشتق ذي رتبة واحدة تؤدي إلى توفير في زمن الحسابات ومتطلّبات الذاكرة . تمّ عرض تطبيق لتحليل ديناميّة الاستقراريّة لنظام القدرة حيث إنّ الصيغ الجديدة لحساسية القيمة المميّزة استخدمت بنجاح لتقدير تأثير تغيّر القيم المتغيّرة على حالات النظام الديناميّة .