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Shock Models with NBRUE and HNBRUE properties

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Abstract. The NBRUE and HNBRUE life distributions are studied in relation to a shock model where shocks are arriving according to a nonhomogeneous Poisson process. The Laplace transform characterizations for these distributions are established. The generating functions for the discrete versions are derived and their physical shock model interpretations are also given.

Introduction

Consider the survival function $\overline{H}$ of a device subjected to shocks. Suppose the probability that the device survives k shocks is $\overline{P}_k; k = 0,1,2,\cdots$ where $1 = \overline{P}_0 > \overline{P}_1 \geq \overline{P}_2 \geq \cdots$. Let $N = \{N(t): \ t \geq o\}$ be the counting process according to which the shocks arrive. The probability $\overline{H}(t)$ that the device survives beyond t is given by

$$\overline{H}(t) = \sum_{k=0}^{\infty} P[N(t) = k]\overline{P}_k \tag{1}$$

Shock models of the type given by (1) are of special interest to the reliability engineers. They would like to study the ageing properties of the survival $\overline{H}(t)$ for components or systems, when some information is available about the magnitue and occurrence of shocks. In biomedicine, $\overline{H}(t)$ can be the survival of an organ which is exposed to environmental or laboratory stress in the form of shocks. Likewise, in risk analysis, claims payable by insurance company represent the shocks. and $\overline{H}(t)$ is the probability that these claims are met by time t.

Shock models of this kind are discussed by various authors. Esary et al. [1] considered the case when N is a homogeneous Poisson process while A. Hameed and Proschan [2;3] discussed the model when N is a nonhomogeneous Poisson or birth process. Block and Savits [4] discussed the model when N is a more general counting process. These authors have proved that $\overline{H}(t)$ is IFR, IFRA, DMRL, NBU or NBUE under suitable conditions on N if $\left[\overline{P}_k\right]_{k=0}^{\infty}$ has the corresponding discrete ageing property. Klefsjo [5] has studied these models for the HNBUE property. Note that IFR $\Rightarrow$ IFRA $\Rightarrow$ NBUE $\Rightarrow$ HNBUE and IFR $\Rightarrow$ DMRL $\Rightarrow$ NBUE are the main implications between these classes of life distributions. Abouammoh *et al.* [6] have studied this model for NBUFR (new better than used in failure rate) and NBAFR (new better than average failure rate) properties while Abouammoh and Hendi [7] have discussed it for NBURFR (new better than used in renewal failure rate) and NBARFR (new better than average renewal failure rate) properties. Also note that NBU $\Rightarrow$ NBUFR $\Rightarrow$ NBAFR.

In this paper, the properties of NBRUE (new better than renewal used in expectation) and HNBRUE (harmonic new better than renewal used in expectation) are studied when N is a nonhomogenous Possion process. Conditions are established under which the NBRUE and HNBRUE properties of $\overline{P}_k$ are preserved for $\overline{H}(t)$ under the shock model (1). The Laplace transform characterizations of these life distributions are presented and the generating functions of the corresponding discrete versions are also discussed.

In 2nd section we present the definitions of the NBRUE and HNBRUE classes of life distributions. The survival function $\overline{H}(t)$ is studied when shocks arrive according to a nonhomogeneous Poisson processes in 3rd section. The Laplace transorms of NBRUE and HNBRUE are discussed in 4th section and the generating functions of these distributions are presented in 5th section.

The NBRUE and HNBRUE Classes of Life Distributions

Suppose that F(t) is the cumulative life distribution function that is F(–0) = 0 and $\overline{F}(t) = 1 - F(t)$, the corresponding survival function. Consider a device with lifelength T and life distribution F(t). The device is replaced instantly upon failure by a sequence of mutually independent devices. These devices are independent of the first unit and identically distributed with the same distribution F. In the long run, when the renewal of the system is continued indefinitely, the remaining life distribution of a device in operation at time t is

$W_F(t) = \int_0^t \overline{F}(u)\ du / \mu_F ; t \geq 0$ where $\mu_F = \mu = \int_0^\infty \overline{F}(u)\ du$, see Barlow and Proschan [8].

It can be seen that $r_w(t) = 1/\mu_F(t)$; where $r_w(t)$ is the failure rate of the renewal (or the equilibrium) distribution W(t). The renewal distribution and its comparison with the parent (or original) life distribution arise in construction of various preventive or repair maintenance policies, see Barlow and Proschan [8]. This has motivated Abouammoh *et al.* [9] to introduce the following classes of life distributions.

Definition (I)

A life distribution F and its survival function $\overline{F}$ are said to be NBRUE (new better than renewal used in expectation) if

$$\int_t^\infty \int_x^\infty \overline{F}(u)dudx \le \mu \int_t^\infty \overline{F}(u)du; \quad t, x \ge 0 \tag{2}$$

where $\mu = \int_0^\infty \overline{F}(u)du$ is the finite mean. The relationship (2) can be expressed as

$$\mu_w(t) \le \mu; \quad t \ge 0 \tag{3}$$

where $\mu_w(t)$ is the mean remaining life of the renewal distribution of F and this accounts for the name NBRUE. The dual NWRUE (new worse than renewal used in expectation) can be defined by reversing the inequality in (2) and (3).

Definition (II)

A life distribution on $(0,\infty)$ with $F(x) = 0$ for $x < 0$, is said to have the harmonic new better than renewal used in expectation (HNBRUE) property if

$$\int_t^\infty \int_x^\infty \overline{F}(u)dudx \le e^{-t/\mu} \int_0^\infty \int_x^\infty \overline{F}(u)dudx, \quad t > 0 \tag{4}$$

with a little algebraic manipulation, (4) can be expressed as

$$\frac{1}{t}\int_0^t \frac{1}{\mu_w(y)}dy \ge \frac{1}{\mu} \tag{5}$$

and this accounts for the name HNBRUE. The dual HNWRUE (harmonic new worse than renewal used in expectation) cab be defined by reversing the inequalities in (4) or (5). Any life distribution F is both NBRUE and NWRUE or both HNBRUE and HNWRUE if it is exponential. For the properties of NBRUE and HNBRUE classes and their relationship with other life distributions one may refer to Abouammoh *et al.* [9]. Cao and Wang [10] have studied the NBRU property under the name new better than used in convex ordering (NBUC).

It may be noted from [9] that the NBRUE and HNBRUE classes of life distributions or their duals are larger than most existing classes of life distributions. For instance note that NBU ⇒ NBUE ⇒ HNBUE ⇒ HNBRUE and NBUE ⇒ NBRUE ⇒ HNBRUE.

For the study on shock models, we need the discrete forms of these life distributions and we define them as follows.

Definition (III)

A life distribution P_k or its survival function $\overline{P}_k = 1 - P_k$; $k = 0, 1, 2, \cdots$ is called NBRUE (NWRE) if

$$\sum_{j=k}^{\infty}\sum_{i=j}^{\infty}\overline{P}_i \le (\ge)\ \mu\sum_{j=k}^{\infty}\overline{P}_j \qquad ;k=0,1,2,\cdots \tag{6}$$

where $\mu=\sum_{j=0}^{\infty}\overline{P}_j$

Definition (IV)

A life distribution P_k or its survival function $\overline{P}_k$, $k=0,1,2,\cdots$ is called HNBRUE (HNWRUE) if

$$\sum_{j=k}^{\infty}\sum_{i=j}^{\infty}\overline{P}_i \le (\ge)\left(1-\frac{1}{\mu}\right)^k\sum_{j=0}^{\infty}\sum_{i=j}^{\infty}\overline{P}_j \qquad ;k=0,1,2,\cdots \tag{7}$$

Nonhomogeneous Poisson Shock Model

Here we discuss the shock model given by (1) when the shocks arrive according to a nonhomogeneous Posisson process with mean value function $\Delta(t)$ and event rate $\lambda(t)=\frac{d\Delta(t)}{dt}$, both defined over $(0,\infty)$. $\lambda(0)=\Delta'(0)$is taken as the right derivative of $\Delta(t)$ at $t=0$. In this set up, the shock mode (1) can be written as

$$\overline{H}(t)=\sum_{k=0}^{\infty}\frac{e^{-\Delta(t)}\Delta^k(t)}{k!}\ \overline{P}_k \tag{8}$$

Now, we investigate the conditions under which NBRUE and HNBRUE properties of $\overline{P}_k$ are preserved in $\overline{H}(t)$ under the model (8). For this, we state the following theorem.

Theorem (I)

The survival $\overline{H}(t)$ in model (8) is NBRUE if $[P_k]_{k=0}^{\infty}$is discrete NBRUE, $\Delta'(0)\neq 0$, $\Delta'(\infty)=\infty$, $t\ge 0$ and

$$(1-\mu)^{-1}\sum_{j=0}^{\infty}[\mu-(j+2)]\,\overline{H}_{j+1}(t)\le[1-\mu\Delta'(t)]^{-1}\sum_{j=0}^{\infty}[\mu\Delta'(t)-(j+2)]\overline{H}_{j+1}(t)$$

where

$$\overline{H}_{j+1}(t)=\sum_{k=0}^{\infty}\frac{e^{-\Delta(t)}\Delta^k(t)}{k!}\overline{P}_{k+j+1} \tag{9}$$

Proof

The survival function $\overline{P}_k$ has the discrete NBRUE properties if

$$\sum_{j=k}^{\infty}\sum_{i=j}^{\infty}\overline{P}_i \leq \mu\sum_{j=k}^{\infty}\overline{P}_j \qquad ;k=0,1,2,\cdots \tag{6}$$

Applying the summations. this can be written as

$$\overline{P}_k + 2\overline{P}_{k+1} + 3\overline{P}_{k+2}\cdots \leq \mu\left[\overline{P}_k + \overline{P}_{k+1} + \cdots\right],$$

i.e.
$$(1-\mu)\overline{P}_k \leq (\mu-2)\overline{P}_{k+1} + (\mu-3)\overline{P}_{k+2} + \cdots$$

Multiplying both sides by the kernel $\frac{e^{-\Delta(t)}\Delta^k(t)}{k!}$ and taking the summation over k=0,1,...., we get

$$(1-\mu)\sum_{k=0}^{\infty}\frac{e^{-\Delta(t)}\Delta^k(t)}{k!}\ \overline{P}_k \leq (\mu-2)\sum_{k=0}^{\infty}\frac{e^{-\Delta(t)}\Delta^k(t)}{k!}\ \overline{P}_{k+1} + \cdots$$

and writing $\sum_{k=0}^{\infty}\frac{e^{-\Delta(t)}\Delta^k(t)}{k!}\ \overline{P}_k$ as $\overline{H}(t)$ from (8), we get

$$(1-\mu)\ \overline{H}(t) \leq (\mu-2)\sum_{k=0}^{\infty}\frac{e^{-\Delta(t)}\Delta^k(t)}{k!}\ \overline{P}_{k+1} + (\mu-3)\sum_{k=0}^{\infty}\frac{e^{-\Delta(t)}\Delta^k(t)}{k!}\ \overline{P}_{k+2} + \cdots$$

i.e.
$$(1-\mu)\ \overline{H}(t) \leq \sum_{j=0}^{\infty}\left[\mu-(2+j)\right]\ \overline{H}_{j+1}(t),$$

or
$$\overline{H}(t) \leq \frac{1}{(1-\mu)}\sum_{j=0}^{\infty}\left[\mu-(2+j)\right]\ \overline{H}_{j+1}(t), \tag{10}$$

Assume that $\overline{H}(t)$ is NBRUE, then

$$\int_t^{\infty}\int_x^{\infty}\overline{H}(u)\ du dx \leq \mu\int_t^{\infty}H(u)\ du$$

Replacing $\overline{H}(u)$ on both sides by the kernel $\sum_{k=0}^{\infty}\frac{e^{-\Delta(t)}\Delta^k(t)}{k!}\ \overline{P}_k$, we get

$$\int_t^{\infty}\int_x^{\infty}\sum_{k=0}^{\infty}\frac{e^{-\Delta(u)}\Delta^k(u)}{k!}\overline{P}_k du dx \leq \mu\int_t^{\infty}\sum_{k=0}^{\infty}\frac{e^{-\Delta(u)}\Delta^k(u)}{k!}\overline{P}_k du$$

which can be written as

$$\sum_{k=0}^{\infty}\int_t^{\infty}\overline{P}_k\left[\int_x^{\infty}\frac{e^{-\Delta(u)}\Delta^k(u)}{k!}\frac{d\Delta(u)}{\Delta'(u)}\right]dx \leq \mu\sum_{k=0}^{\infty}\overline{P}_k\int_t^{\infty}\frac{e^{-\Delta(u)}\Delta^k(u)}{k!}\frac{d\Delta(u)}{\Delta'(u)}$$

By applying the 2nd mean value theorem we have

$$\sum_{k=0}^{\infty}\int_t^{\infty}\overline{P}_k\frac{1}{\Delta'(x)}\left[\int_x^{\infty}\frac{e^{-\Delta(u)}\Delta^k(u)}{k!}d\Delta(u)\right]dx \leq \mu\sum_{k=0}^{\infty}\overline{P}_k\frac{1}{\Delta'(t)}\left[\int_t^{\infty}\frac{e^{-\Delta(u)}\Delta^k(u)}{k!}d\Delta(u)\right]$$

i.e.

$$\sum_{k=0}^{\infty}\overline{P}_k\int_t^{\infty}\frac{1}{\Delta'(x)}\left[\sum_{j=0}^{k}\frac{e^{-\Delta(x)}\Delta^j(x)}{j!}\right]dx \le \mu\sum_{k=0}^{\infty}\overline{P}_k\,\frac{1}{\Delta'(t)}\sum_{j=0}^{k}\frac{e^{-\Delta(t)}\Delta^j(t)}{j!}$$

or

$$\sum_{k=0}^{\infty}\overline{P}_k\sum_{j=0}^{k}\left[\int_t^{\infty}\frac{e^{-\Delta(x)}\Delta^j(x)}{j!}\,\frac{d\Delta(x)}{\Delta'^2(x)}\right]\le\frac{\mu}{\Delta'(t)}\sum_{j=0}^{\infty}\frac{e^{-\Delta(t)}\Delta^j(t)}{j!}\sum_{k=j}^{\infty}\overline{P}_k$$

i.e.

$$\sum_{k=0}^{\infty}\frac{1}{\Delta'^2(t)}\sum_{i=0}^{j}\frac{e^{-\Delta(t)}\Delta^i(t)}{i!}\sum_{k=j}^{\infty}\overline{P}_k\le\frac{\mu}{\Delta'(t)}\sum_{j=0}^{k}\frac{e^{-\Delta(t)}\Delta^j(t)}{j!}\left[\overline{P}_j+\overline{P}_{j+1}+\cdots\right]$$

Some further simplifications give

$$\left(\frac{1}{\Delta'(t)}-\mu\right)\sum_{j=0}^{\infty}\frac{e^{-\Delta(t)}\Delta^j(t)}{j!}\,\overline{P}_j\le\sum_{j=0}^{\infty}\Bigg[\left(\mu-\frac{2}{\Delta'(t)}\right)\frac{e^{-\Delta(t)}\Delta^j(t)}{j!}\,\overline{P}_{j+1}$$
$$+\left(\mu-\frac{3}{\Delta'(t)}\right)\frac{e^{-\Delta(t)}\Delta^j(t)}{j!}\,\overline{P}_{j+2}+\cdots\Bigg]$$

or

$$\frac{(1-\mu\Delta'(t))}{\Delta'(t)}\overline{H}(t)\le\frac{1}{\Delta'(t)}\sum_{j=0}^{\infty}\left[(\mu\Delta'(t)-2)\frac{e^{-\Delta(t)}\Delta^j(t)}{j!}\,\overline{P}_{j+1}+\cdots\right]$$

or

$$\overline{H}(t)\le\frac{1}{(1-\mu\Delta'(t))}\sum_{j=0}^{\infty}[\mu\Delta'(t)-(j+2)]\;\overline{H}_{j+1}(t) \tag{11}$$

where $\overline{H}_{j+1}(t)$ is defined in (9)

From (10) and (11), we conclude that $\overline{H}(t)$ satisfies the assumption of NBRUE property if

$$\frac{1}{(1-\mu)}\sum_{j=0}^{\infty}[\mu-(j+2)]\;\overline{H}_{j+1}(t)\le\frac{1}{(1-\mu\Delta'(t))}\sum_{j=0}^{\infty}[\mu\Delta'(t)-(j+2)]\;\overline{H}_{j+1}(t)$$

where $\overline{H}_{j+1}(t)$ is defined in (9). This completes the proof of the theorem.

For the dual property NWRUE, we present the following theorem whose proof can be carried out along the same lines.

Theorem (II)

The survival function $\overline{H}(t)$ in (8) is NWRUE if $\left[P_k\right]_{k=0}^{\infty}$ is discrete NWRUE,

$\Delta'(t)\neq 0,\ \Delta'(\infty)=\infty,\ t\geq 0$,and the condition (8) is satisfied with the inequality sign reversed.

Now, we give a similar theorem for HNBRUE distribution as follows.

Theorem (III)

The survival distribution $\overline{H}(t)$ in model (8) is HNBRUE if $\left[P_k\right]_{k=0}^{\infty}$ is discrete HNBRUE, $\Delta'(0)\neq 0,\ \Delta'(\infty)=\infty$, and

$$e^{-\frac{\Delta(t)}{\mu}}\sum_{\ell=0}^{\infty}(\ell+1)\overline{P}_{\ell}\leq\frac{\Delta'^{2}(t)}{\Delta'^{2}(0)}e^{-t/\mu}\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}P_{k} \tag{12}$$

Proof

The survival function $\overline{P}_k$ has the discrete HNBRUE property if

$$\sum_{j=k}^{\infty}\sum_{i=j}^{\infty}\overline{P}_i\leq\left(1-\frac{1}{\mu}\right)^{k}\sum_{j=0}^{\infty}\sum_{i=j}^{\infty}\overline{P}_i \qquad ;k=0,1,2,\ \cdots$$

Expanding the summation and collecting the terms, this yields

$$\overline{P}_k+2\overline{P}_{k+1}+3\overline{P}_{k+2}+\ \cdots\leq\left(1-\frac{1}{\mu}\right)^{k}\left[\overline{P}_0+2\overline{P}_1+3\overline{P}_2+\ \cdots\ +k\overline{P}_{k-i}+(k+1)\overline{P}_k+\ \cdots\right]$$

Multiplying both sides by the kernel $\sum_{k=0}^{\infty}\frac{e^{-\Delta(t)}\Delta^{k}(t)}{k!}$, we obtain

$$\begin{aligned}\overline{H}(t)+\sum_{\ell=0}^{\infty}(\ell+2)\ \overline{H}_{\ell}(t)&\leq\sum_{k=0}^{\infty}\left(1-\frac{1}{\mu}\right)^{k}\left[\overline{P}_0+2\overline{P}_1+\ \cdots\right]\frac{e^{-\Delta(t)}\Delta^{k}(t)}{k!}\\&=\sum_{k=0}^{\infty}\left(1-\frac{1}{\mu}\right)^{k}\sum_{\ell=0}^{\infty}(\ell+1)\overline{P}_{\ell}\,\frac{e^{-\Delta(t)}\Delta^{k}(t)}{k!}\\&=\sum_{\ell=0}^{\infty}(\ell+1)\overline{P}_{\ell}\sum_{k=0}^{\infty}\left(1-\frac{1}{\mu}\right)^{k}\frac{e^{-\Delta(t)}\Delta^{k}(t)}{k!}\\&=\sum_{\ell=0}^{\infty}(\ell+1)\overline{P}_{\ell}\ \ e^{-\Delta(t)/\mu}\end{aligned}$$

i.e.

$$\overline{H}(t)\leq e^{-\Delta(t)/\mu}\sum_{\ell=0}^{\infty}(\ell+1)\overline{P}_{\ell}-\sum_{\ell=0}^{\infty}(\ell+2)\overline{H}_{\ell}(t) \tag{13}$$

$$= e^{-t/\mu}\int_0^{\infty}\sum_{k=0}^{\infty}\frac{1}{\Delta'(x)}\frac{e^{-\Delta(x)}\Delta^k(x)}{k!}\ \sum_{k=j}^{\infty}\overline{P}_k\ dx$$

$$= e^{-t/\mu}\sum_{j=0}^{\infty}\frac{1}{\Delta'^{\,2}(0)}\left[\int_0^{\infty}\frac{e^{-\Delta(x)}\Delta^j(x)}{j!}\ d\Delta(x)\right]\sum_{k=j}^{\infty}\overline{P}_k\ dx$$

$$= e^{-t/\mu}\frac{1}{\Delta'^{\,2}(0)}\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\overline{P}_k\ . \tag{16}$$

From (15) and (16), we obtain

$$\frac{1}{\Delta'^{\,2}(0)}\left[\overline{H}(t)+\sum_{\ell=0}^{\infty}(\ell+2)\overline{H}_\ell(t)\right]\le\frac{e^{-t/\mu}}{\Delta'^{\,2}(0)}\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\overline{P}_k$$

or

$$\overline{H}(t)\le\frac{\Delta'^{\,2}(t)}{\Delta'^{\,2}(0)}e^{-t/\mu}\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\overline{P}_k\ -\ \sum_{\ell=0}^{\infty}(\ell+2)\overline{H}_\ell(t) \tag{17}$$

From (13) and (17), we obtain

$$e^{-\frac{\Delta(t)}{\mu}}\sum_{\ell=0}^{\infty}(\ell+1)\overline{P}_\ell\le\frac{\Delta'^{\,2}(t)}{\Delta'^{\,2}(0)}e^{-t/\mu}\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\overline{P}_k$$

and this completes the proof.

For the dual property HNWRUE, we state the following theorem which can be proved in similar fashion.

Theorem (IV)

The survival function $\overline{H}(t)$ in (8) is HBWRUE if $\left[\overline{P}_k\right]_{k=0}^{\infty}$ HNWRUE, $\Delta'(0)\neq 0$, $\Delta'(\infty)=\infty$, $t\ge 0$, and the condition (12) is satisfied with the inequality sign reversed.

Laplace Transform

Here we investigate the necessary and sufficient conditions for a life distribtion to have NBRUE and HNBRUE properties in terms of Laplace transforms. These conditons cab be used to investigate closure properties under convolution.

Let F be the distribution function with F(- 0) =0 and let

$$\phi(s)=\int_0^{\infty}e^{-su}dF(u)\qquad ;\ s\ge 0$$

be the Laplace transform of $F'(x)=\frac{d}{dx}F(x)$. Define

$$a_n(s)=\frac{(-1)^n}{n!}\frac{d^n}{ds^n}\left(\frac{1-\phi(s)}{s}\right) \qquad ;\ n\geq 0,\ s\geq 0 \tag{18}$$

Let $\alpha_{n+1}(s)=s^{n+1}a_n(s)$ for $n \geq 0$ and $s \geq 0$ and $\alpha_0(s)=1$ for $s \geq 0$. The transforms $a_n(s)$ and $\alpha_n(s)$ can be written in the forms

$$a_n(s)=\frac{1}{n!}\int_0^\infty u^n e^{-su}\overline{F}(u)\ du \tag{19}$$

and

$$\alpha_{n+1}(s)=\frac{1}{n!}\int_0^\infty s(su)^n e^{-su}\overline{F}(u)\ du \qquad ;n\geq 0,\ s\geq 0 \tag{20}$$

Vinogradov [11] has characterized the IFR property in terms of $\alpha_n(s)$ while block and Savits [4] have obtained similar characterizations for IFRA, DMRL, NBU and NBUE properties. Abouammoh *et al.* [6] have characterized the NBUFR and NBAFR properties similarly. Here we establish similar characterization for the NBRUE and HNRUE properties.

Theorem (V)

Let F be a life distribution with F(- 0) = 0, then.

(i) F has the NBRUE property if and only if (iff)

$$\sum_{j=k}^{\infty}\sum_{i=j}^{\infty}\alpha_{i+1}\leq \mu s\sum_{j=k-1}^{\infty}\alpha_{j+1}, \tag{21}$$

(ii) F has the HNBRUE property iff

$$\sum_{j=k}^{\infty}\sum_{i=j}^{\infty}\alpha_{i+1}\leq \frac{s^2}{\left(1+\frac{1}{\mu s}\right)^{k-1}}\cdot\frac{1}{2}E\left(U^2\right) \tag{22}$$

where U is a positive random variable.

Proof

(i) Assume for the necessary condition that F is NBRUE. Then the form (20) gives

$$\begin{aligned}\sum_{j=k}^{\infty}\sum_{i=j}^{\infty}\alpha_{i+1} &=\sum_{j=k}^{\infty}\sum_{i=j}^{\infty}\int_0^\infty \frac{se^{-su}(su)^i}{i!}\overline{F}(u)du\\ &=s\sum_{j=k}^{\infty}\int_0^\infty\left[\int_0^u \frac{e^{-sy}(sy)^{j-1}}{(j-1)!}d(sy)\right]\overline{F}(u)du\\ &=s^3\int_0^\infty \frac{e^{-st}(st)^{k-2}}{(k-2)!}dt\int_t^\infty\int_y^\infty \overline{F}(u)dudy\end{aligned}$$

$$\le s^3\int_0^\infty \frac{e^{-st}(st)^{k-2}}{(k-2)!}dt \ \ \mu\int_t^\infty \overline{F}(u)du,$$

Using the NBRUE property

$$=\mu s^2\int_0^\infty \overline{F}(u)du\int_0^u \frac{se^{-st}(st)^{k-2}}{(k-2)!}dt,$$

$$=\mu s\sum_{j=k-1}^\infty \int_0^\infty \frac{se^{-su}(su)^j}{j!}\overline{F}(u)du$$

$$=\mu s\sum_{j=k-1}^\infty \alpha_{j+1}.$$

Thus we get the necessary conditon as

$$\sum_{j=k}^\infty\sum_{i=j}^\infty \alpha_{i+1}\le \mu \ \ s\sum_{j=k-1}^\infty \alpha_{j+1} \tag{23}$$

Now, we show the sufficiency of this conditon. Equation (23) states

$$\sum_{j=n}^\infty\sum_{i=j}^\infty\int_0^\infty \frac{S^{i+1}u^ie^{-su}}{i!}\overline{F}(u)du\le \mu \ \ s\sum_{j=n-1}^\infty\int_0^\infty \frac{s^{j+1}u^je^{-su}}{j!}\overline{F}(u)du$$

that is

$$\int_0^\infty\sum_{j=n}^\infty \frac{(sy)^{j-1}e^{-sy}}{(j-1)!}\left(\int_y^\infty \overline{F}(u)du\right)dy\le \ \mu\int_0^\infty\sum_{j=n}^\infty \frac{(sy)^{j-1}e^{-sy}}{(j-1)!}\overline{F}(u)dy$$

or

$$\int_0^\infty G_n(y)\int_y^\infty \overline{F}(u)du \ \ dy\le \ \mu\int_0^\infty G_n(y) \ \ \overline{F}(y)dy,$$

where

$$G_n(y)=\sum_{j=n}^\infty \frac{(sy)^{j-1}e^{-sy}}{(j-1)!}$$

substituting $s = n/t$, we have

$$G_n(y)=\sum_{j=n}^\infty \frac{((n/t)y)^{j-1}e^{-(n/t)y}}{(j-1)!}$$

which means that $G_n(\cdot)$ is a gamma distribution with characteristic function

$$\phi_n(w)=\left(1-\frac{iwt}{n}\right)^{-n}$$

but

$$\lim_{n\to\infty}\phi_n(w)=\exp(iwt)$$

which implies that $G_n(\cdot)$ converges to the degenerate distribution

$$G(x)=\begin{cases}0 & \cdots \text{ for } x<t\\ 1 & \cdots \text{ for } x\geq t.\end{cases}$$

This means that

$$\lim_{n\to\infty}\int_0^\infty G_n(y)\left(\int_y^\infty \overline{F}(u)du\right) dy \leq \mu \lim_{n\to\infty}\int_0^\infty G_n(y)\ \overline{F}(y)\ dy$$

or that

$$\int_t^\infty\int_y^\infty \overline{F}(u)du\ dy \leq \mu\int_t^\infty \overline{F}(y)\ dy$$

which means that $\overline{F}(u)$has the NBRUE property. This completes the proof of the part (i).

(ii) Again for the necessary condition of this part, we assume that F is HNBRUE. Then, we have

$$\sum_{j=k}^\infty\sum_{i=j}^\infty \alpha_{i+1}=\sum_{j=k}^\infty\sum_{i=j}^\infty\int_0^\infty \frac{s(su)^i e^{-su}}{i!}\ \overline{F}(u)\ du$$

By rearrangement of integrals and summations, this can be written as

$$\sum_{j=k}^\infty\sum_{i=j}^\infty \alpha_{i+1}=s\sum_{j=k}^\infty\int_0^\infty \frac{se^{-sy}(sy)^{j-1}}{(j-1)!}\ dy\int_y^\infty \overline{F}(u)\ du$$

$$=s^3\int_0^\infty \frac{e^{-st}(st)^{k-2}}{(k-2)!}\ dt\int_t^\infty\int_y^\infty \overline{F}(u)\ du\ dy$$

$$\leq s^3\int_0^\infty \frac{e^{-st}(st)^{k-2}}{(k-2)!}\ dt\ \ e^{-t/\mu}\int_0^\infty\int_y^\infty \overline{F}(u)\ du\ dy,$$

by the HNBRUE condition and this on further simplification can be written as

$$\sum_{j=k}^\infty\sum_{i=j}^\infty \alpha_{i+1}=\leq s^3\int_0^\infty \frac{e^{-st(1+(1/\mu s))}(st)^{k-2}}{(k-2)!}\ dt\int_0^\infty\int_y^\infty \overline{F}(u)\ dudy,$$

$$=\frac{s^2}{\left(1+\frac{1}{\mu s}\right)^{k-1}}\int_0^\infty \frac{e^{-st(1+(1/\mu s))}\left[st(1+(1/\mu s))\right]^{k-2}s(1+(1/\mu s))}{(k-2)!}\ dt\int_0^\infty\int_y^\infty \overline{F}(u)dudy,$$

$$=\frac{s^2}{\left(1+\frac{1}{\mu s}\right)^{k-1}}\int_0^\infty \overline{F}(u)\ \int_0^u dydu$$

$$= \frac{s^2}{\left(1+1/\mu s\right)^{k-1}} \int_0^\infty u\bar{F}(u)\ du$$

$$= \frac{s^2}{\left(1+1/\mu s\right)^{k-1}} \cdot \frac{1}{2} E\left[U^2\right],$$

where U is a positive random variable, and this proves the necessary condition.

For the sufficiency of this condition, we observe that

$$\sum_{j=k}^\infty \sum_{i=j}^\infty \alpha_{i+1} \le \frac{s^2}{\left(1+1/\mu s\right)^{k-1}} \cdot \frac{1}{2} E\left[U^2\right],$$

implies

$$\sum_{j=k}^\infty \sum_{i=j}^\infty \int_0^\infty \frac{s^{i+1}u^i e^{-su}}{i!} \bar{F}(u)du \le \frac{s^2}{\left(1+1/\mu s\right)^{n-1}} \int_0^\infty u\bar{F}(u)\ du,$$

or

$$s^2 \sum_{j=n}^\infty \int_0^\infty \int_0^u \frac{(sy)^{j-1} e^{-su}}{(j-1)!}\ dy\ \ \bar{F}(u)du \le \frac{s^2}{\left(1+1/\mu s\right)^{n-1}} \int_0^\infty \int_y^\infty \bar{F}(u)du\ \ dy.$$

Further rearrangement of integrals and summations yields

$$\int_0^\infty \left(\sum_{j=n}^\infty \frac{(sy)^{j-1} e^{-sy}}{(j-1)!} \right) \left(\int_y^\infty \bar{F}(u)du \right) dy \le \frac{s^2}{\left(1+1/\mu s\right)^{n-1}} \int_0^\infty \int_y^\infty \bar{F}(u)dudy.$$

or that

$$\int_0^\infty G_n(y) \int_y^\infty \bar{F}(u)du\ \ dy \le \frac{1}{\left(1+1/\mu s\right)^{n-1}} \int_0^\infty \int_y^\infty \bar{F}(u)dudy.$$

where

$$\int_0^\infty G_n(y) = \sum_{j=n}^\infty \frac{(sy)^{j-1} e^{-su}}{(j-1)!}$$

and the substitution s = t/n yields

$$G_n(y) = \sum_{j=n}^\infty \frac{\left[(t/n)y\right]^{j-1} e^{-(t/n)y}}{(j-1)!},$$

which means that $G_n(\cdot)$ is a gamma distribution with characteristic function

$$\phi_n(w) = \left(1 - \frac{iwt}{n}\right)^{-n}, \text{ with } \lim_{n\to\infty} \phi_n(w) = e^{iwt}.$$

Thus $G_n(\cdot)$ converges to the degenerate distribution

$$G(x) = \begin{cases} 0 & \text{for } x < t \\ 1 & \text{for } x \ge t. \end{cases}$$

This implies that

$$\lim_{n\to\infty}\int_0^\infty G_n(y)\left(\int_y^\infty \overline{F}(u)du\right)\ dy \le \lim_{n\to\infty}\frac{1}{\left(1+{}^{t}\!/\!_{n\mu}\right)^{n-1}}\int_0^\infty\int_y^\infty \overline{F}(y)\ dy$$

or

$$\int_0^\infty\int_y^\infty \overline{F}(u)dudy \le e^{-t/\mu}\int_0^\infty\int_y^\infty \overline{F}(u)dudy.$$

and this proves the sufficiency condition.

similar conditions in terms of α and $E\left[U^2\right]$ hold for NWRUE and HNWRUE classes and can be proved by reversing all previous inequalities in Theorem (v).

Generating Functions

Suppose that $\overline{P}_j$ is the survival probability of a nonnegative random variable X. Let

$$\psi(\theta)=E\left(\theta^j\right)=\sum_{j=0}^{\infty}\theta^j\ \ p_j$$

be the corresponding generating function. then $\psi(\theta)$ can be written as

$$\psi(\theta)=1-\sum_{j=0}^{\infty}(1-\theta)\ \ \theta^j\ \ \overline{P}_j \tag{24}$$

For a geometric random variable X with parameter θ as the probability of survival of a shock, we have

$$P(X=j)=(1-\theta)\ \ \theta^j \qquad ;j=0,1,2,\cdots . \tag{25}$$

This is the probability of surviving j shocks before failure. $\psi(\theta)$ can be written as

$$\psi(\theta)=1-\sum_{j=0}^{\infty}P(X=j)\ \ \overline{P}_j \tag{26}$$

Now let y_i; $i = 1, 2, \ldots, n$ be independently and identically distributed random variables each having a probability distribution given by (25). Then $Z=\sum_{i=1}^{n}y_i$ has a negative binomial distribution with probability mass function (pmf) defined as

$$P[Z=j]=\binom{n+j-1}{j}\ \theta^j\ \ (1-\theta)^n \qquad ;j=0,1,2,\cdots . \tag{27}$$

Next define

$$\beta_n(\theta)=\begin{cases} p^n b_n(\theta) & ,n=1,2,\cdots . \\ 1 & ,n=0 \end{cases} \tag{28}$$

where $p = 1 - q = 1 - \theta$ and

$$b_n(\theta) = \sum_{j=0}^{k} \binom{n+j-1}{j} \theta^j \ \bar{P}_j$$

Further define

$$\beta_{n,\,k}(\theta) = \sum_{j=0}^{k} (1-\theta)^n \binom{n+j-1}{j} \theta^j \ \bar{P}_j \qquad ; k = 0, 1, 2, \cdots \tag{29}$$

The form (29) can be interpretted as follows. Suppose a device is exposed to two types of shocks say type I and type II. In unit time, type I and type II shocks occur with probabilities p and q respectively. If Y_j denotes the number of type I shocks between (j-1)th and jth shock of type II, then Y_j has the geometric distribution with pmf given by (25). Thus $\beta_n(p)$ is the probablity that the device survives until k shocks of type I given that the probability of surviving shocks of type II is $\bar{P}_j$.

Using the form (29), Abouammoh and Hendi [12] have found conditons under which a generating function is the generating function of an IFR, IFRA, NBU, NBUFR or NBAFR life distribution. Again Abouammoh and Hendi [7] have similarly found conditions for NBURFR and NBARFR classes.

Here, we find conditions on generating functions for a life distribution to be NBRUE or HNBRUE. In this regard, we state the following theorem.

Theorem (V)

Let $B_n(\theta)$ and $B_{n,k}(\theta)$ be given by (28) and (29) respectively. Then

(i) $\bar{P}_k$ is NBRUE if

$$\beta_n(p) \geq \sum_{j=0}^{\infty} \left(\frac{j+2-\mu}{\mu-1} \right) \beta_{n,j}(p) \qquad ; n = 1, 2, \cdots$$

where

$$\beta_{n,j}(p) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} p^n \ q^k \ \bar{P}_{k+j+1}$$

(ii) $\bar{P}_k$ is HNBRUE if

$$B_n(p) \leq \sum_{j=0}^{k-1} \left[\left(\sum_{\ell=0}^{\infty} \bar{P}_{j+\ell} \right) \binom{n+j-1}{j} p^n q^j \right] \Big/ \left[1 - (1 - 1/\mu)^k \right] - \sum_{k=0}^{\infty} B_{n,k}(p)$$

where

$$B_{n,k}(p) = \sum_{j=0}^{\infty} \binom{n+j-1}{j} p^n q^j \ \bar{P}_{k+j+1} \quad ; \ k = 0, 1, 2, \cdots$$

Proof (i)

The life distribution $\overline{P}_k$, $k = 0,1,2,\ldots$ is NBRUE if

$$\sum_{j=k}^{\infty}\sum_{i=j}^{\infty}\overline{P}_i \le \mu\sum_{j=k}^{\infty}\overline{P}_j \quad ;k = 0,1,2,\cdots . \tag{30}$$

This can be written as

$$\sum_{j=k}^{\infty}\left[\overline{P}_j + \overline{P}_{j+1} + \cdots\right] \le \mu\left[\overline{P}_k + \overline{P}_{k+1} + \cdots\right]$$

Applying the summation over j and simplifying, we get

$$(2-\mu)\ \overline{P}_{k+1} + (3-\mu)\ \overline{P}_{k+2} + \cdots \le (\mu - 1)\ \overline{P}_k, \tag{31}$$

or

$$\sum_{j=0}^{\infty}[j+2-\mu]\ \overline{P}_{j+k+1} \le (\mu-1)\ \overline{P}_k \tag{32}$$

Multiplying both sides by the kernel

$$\sum_{k=0}^{\infty}\binom{n+k-1}{k}\ p^n\ q^k,$$

we get

$$\sum_{j=0}^{\infty}(j+2-\mu)\sum_{k=0}^{\infty}\binom{n+k-1}{k}\ p^n\ q^k\ \overline{P}_{j+k+1} \le (\mu-1)\sum_{k=0}^{\infty}\binom{n+k-1}{k}\ p^n\ q^k\ \overline{P}_k$$

or

$$\sum_{j=0}^{\infty}(j+2-\mu)\ B_{n,j}(p) \le (\mu-1)\ B_n(p)$$

or

$$B_n(p) \ge \left[\sum_{j=0}^{\infty}(j+2-\mu)\ B_{n,j}(p)\right] / (\mu-1)$$

This completes the proof.

Proof (ii)

$\overline{P}_k$, $k = 0, 1, 2, \ldots$ is HNBRUE if

$$\sum_{j=k}^{\infty}\sum_{i=j}^{\infty}\overline{P}_i \le \left(1-\frac{1}{\mu}\right)^k \sum_{j=0}^{\infty}\sum_{i=j}^{\infty}\overline{P}_i \quad ;k = 0,1,2,\ \cdots$$

Applying summation over i gives

$$\sum_{j=k}^{\infty}\left[\overline{P}_j + \overline{P}_{j+1} + \cdots\right] \le \left(1-\frac{1}{\mu}\right)^k \sum_{j=k}^{\infty}\left[\overline{P}_j + \overline{P}_{j+1} + \cdots\right]$$

i.e.

$$\sum_{j=k}^{\infty}\left[\overline{P}_j + \overline{P}_{j+1} + \cdots\right] - \sum_{j=0}^{k-1}\left[\overline{P}_j + \overline{P}_{j+1} + \cdots\right] \le \left(1-\frac{1}{\mu}\right)^k \sum_{j=0}^{\infty}\left[\overline{P}_j + \overline{P}_{j+1} + \cdots\right]$$

Multiplying both sides by the kernel $\binom{n+j-1}{j}\ p^n\ \ q^j$, we get

$$\sum_{j=0}^{\infty}\binom{n+j-1}{j}\ p^n q^j \overline{P}_j+\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\binom{n+j-1}{j}\ p^n q^j \overline{P}_{j+k+1}-\sum_{j=0}^{k-1}\binom{n+j-1}{j}\ p^n q^j\ \left[\overline{P}_j+\overline{P}_{j+1}+\cdots\right]$$

$$\leq\left(1-\frac{1}{\mu}\right)^k\left[\sum_{j=0}^{\infty}\binom{n+j-1}{j}\ p^n q^j \overline{P}_j+\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\binom{n+j-1}{j}\ p^n q^j \overline{P}_{j+k-1}\right]$$

or

$$\left[B_n(p)+\sum_{k=0}^{\infty}B_{n.k}(p)\right]-\sum_{j=0}^{k-1}\binom{n+j-1}{j}\ p^n q^j\ \left[\overline{P}_j+\overline{P}_{j+1}+\cdots\right]$$

$$\leq\left(1-\frac{1}{\mu}\right)^k\left[B_n(p)+\sum_{k=0}^{\infty}B_{n,k}(p)\right]$$

This implies

$$\left[1-\left(1-\frac{1}{\mu}\right)^k\right]\left[B_n(p)+\sum_{k=0}^{\infty}B_{n,k}(p)\right]\leq\sum_{j=0}^{k-1}\binom{n+j-1}{j}\ p^n q^j\ \left[\overline{P}_j+\overline{P}_{j+1}+\cdots\right]$$

Hence

$$B_n(p)\leq\left[\sum_{j=0}^{k-1}\left[\sum_{\ell=0}^{\infty}\overline{P}_{j+\ell}\right]\binom{n+j-1}{j}\ p^n q^j\right]\Big/\left[1-\left(1-\frac{1}{\mu}\right)^k\right]-\sum_{k=0}^{\infty}B_{n,k}(p)$$

which completes the proof.

The corresponding results for NWRUE and HNWRUE classes can be proved by similar arguments.

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خواص نماذج الصدمات لفصليْ توزيعات الحياة
NBRUE و HNBRUE

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(سُلِّمَ في ١١/٩/١٤١٢هـ؛ وقُبل للنشر في ٤/٥/١٤١٥هـ).

ملخص البحث. درسنا في هذا البحث فصليْ توزيعات الحياة الجديد أفضل من المستخدم المتجدد في التوقع، والتوافقي الجديد أفضل من المستخدم المتجدد في التوقع، وذلك فيما يتعلق بنماذج الصدمات التي تصل وفقًا لعملية بواسون غير المتجانسة. أوجدنا تمييزًا لهذين الفصلين بوساطة تحويل لابلاس. اشتققنا الدالة المولدة للصيغة المتقطعة لهذين الفصلين وأوضحنا تفسيرًا علميًّا لنموذج الصدمات فيهما.