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Approximation for N-Gluon Scattering

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Abstract. In this paper, we give a simple approximation for n-gluon scattering. This approximation is based on the Parke and Taylor formula to leading order in the number of color and the proportional factor, Pn. Our approximation gives excellent accuracy for the 6-gluon 2->4 process and 7-gluon 2->5 process when compared with exact results. It is shown, using this approximation, that the error is dramatically reduced to about 1%, compared to the exact result for seven gluon scattering.

Introduction

QCD multi-jet production will be an important background to many physics processes that we wish to study at future hadron colliders. Higgs productions, heavy quark (top) pairs, vector boson pairs, all have multi-jet final states as the dominant decay mode. To reliably estimate the background, one will need the cross-section for hard, well-separated jets characteristic of the new physics signal. In such configurations, jet fragmentation Monte Carlos based on leading pole implementations of QCD will not be valid. One will rather need exact tree-level matrix elements for 2->n parton-parton subprocesses.

There are various proposals for estimating 2->n multi-gluon cross-sections. The first proposal was done by Maxwell [1]. His method is based on a reduction scheme, in which the pair of most collinear gluons is replaced by a single on-shell gluon and some collinearity factor. This process continues until only gg $\rightarrow$ ggg are left and used the exact Parke and Taylor formula [2].

For a 2→n Scattering amplitude, where a pair of final particle i.j. becomes collinear, one has denoting Pi.Pj by (i.j).

$$\lim_{(i\cdot f)\to o}(i\cdot j)\left|M_n\right|^2 = g^2 P_{ab}(Z)\left|M_{\pi-1}\right|^2 \tag{1}$$

where g is QCD coupling constant. For a gluon pair becoming collinear ab = gg and $P_{gg}(z)$ is the Altarelli - Parisi splitting Kernel and given by

$$P_{gg}(z) = N_c[z^4 + (1-z)^4 + 1] / z(1-z) \tag{2}$$

To approximate $|M_n|^2$ for some (n-1) jet configuration one has

$$\left|M_n\right|^2 \cong \frac{\left|M_n\right|^2}{\left|M_n^{PT}\right|^2}\Big|_{IR} X \left|M_n^{PT}\right|^2 \tag{3}$$

$|_{IR}$ denotes the ratio of amplitudes is evaluated at a ;nearbyس configuration where two of the final jets i, j, having the smallest invariant mass, have been replaced by a collinear pair of jet in the direction of Pi+Pj with energy fraction z and (1-z). The ratio in equation (3) contains no kinematical poles, it will be reasonably slow-varying. In fact, the Parke and Taylor formula which is given, for general n, by the following form

$$\left|M_n^{PT}\right|^2 = g^{2n-4} \frac{N_c^{n-2}}{(N_c^2-1)} \cdot 2^{4-n} \sum_{i<j}(ij)^4 \sum_P \frac{1}{(123\ldots n)} \tag{4}$$

where , "p" denotes a sum over the (n-1)!/2 distinct permutations of 1,2,3,...n, and the amplitude squared is helicity and color summed and averaged, does not reduce by this method. This means that z4 + $(1 - z)^4$ and one have different coefficients. Defining the ratio of these coefficients to be R,

$$R \equiv \sum_{l,n+i\cdot j} (lm)^4 \Big/ \sum_{l+i\cdot j} (la)^4 \tag{5}$$

one has

$$\lim_{(i\cdot j)\to o}(i\cdot j)\left|M_n^{PT}\right|^2 = g^2 f(R,z)\left|M_{n-1}^{PT}\right|^2 \tag{6}$$

where

$$f(R,z) \equiv N_c[z^4 + (1-z)^4 + R]/(1+R)z(1-z)$$

Taking the ratio of equations (1) and (6), and using equation (3) we find

$$|M_n|^2 \cong F_1(R,z)\frac{|M_{n-1}|^2}{|M_{n-1}^{PT}|^2}\cdot|M_n^{PT}|^2 \tag{7}$$

with

$$F_1(R,z) \equiv (1+R)[z^4 + (1-z)^4 + 1]/[z^4 + (1-z)^4 + R] \tag{8}$$

For six-gluon scattering (n=6): then equation (7) can be written as:

$$|M_6|^2 \cong F_1(R,z)|M_6^{PT}|^2 \tag{9}$$

For seven-gluon scattering (n=7) one has

$$|M_7|^2 \cong F_2(R,z)F_1(R,z)|M_6^{PT}|^2 \tag{10}$$

Equations (9) and (10) are the main result of ref [1].

Still, we have some difficulties in mind. For example, the collinearity factor, which appeared in equations (9) and (10), have been calculated numerically and the factor becomes even more complicated for four and five jets.

The second proposal was suggested by Kunszt and Stirling [3]. This method is based on the assumption that all non-zero helicity amplitudes contribute equally to the matrix element squared. In this method, only the Parke and Taylor helicity amplitudes are evaluated in leading order, in the number of color. The full matrix elements squared is then obtained by multiplication with a combinatorial factor. This means that

$$|M_n|^2 \cong \frac{2^n - 2(n+1)}{n(n-1)}|M_n^{PT}|^2 \tag{11}$$

For n=4 and n=5, the non-leading terms in Parke and Taylor formula are zero, and one-re-obtains the known $2 \to 2$ and $2 \to 3$ results

$$|M_4|^2 = \frac{1}{2}|M_4^{PT}|^2 \tag{12}$$

$$|M_5|^2 = |M_5^{PT}|^2 \tag{13}$$

For n=6, we found by numerical comparison, that $|M_6^{PT}|^2$ is about 60% of the total six-gluon scattering. For seven, gluon scattering $|M_7^{PT}|^2$ is about 40% of the exact results. This means that the Parke and Taylor helicity amplitude is decreasing by increasing the number of the final states gluon.

n-Gluon Approximation

In this paper we shall improve the results which are obtained in ref. [1] and ref. [3]. Our approximation is based on the Parke and Taylor formula, to leading order in the number of color and the proportional factor, Pn. The full result for the matrix element squared is obtained by the leading PT formula multiplication with the proportional factor, then one has

$$|M_n|^2 \cong P_n|M_n^{PT}|^2 \tag{14}$$

This means that we replaced the combinatorial factor equation (1) by the proportional factor, Pn.

Pn is defined as Pn = f''/f where, f'' is the number of the subsets between two gluon interaction in the final states and f is the number of final jets.

For n=4, gg $\rightarrow$ gg, denoting the final states gluon by (1,2) and the subset between two gluon interactions in the final states is {1,2}. Then, we have only one subset and we have two gluon in the final states. From the definition of Pn one has

$$P_4 = \frac{1}{2}$$

Then, the formula which is represented in (14), can be written as

$$|M_4| = P_4|M_4^{PT}|^2 = \frac{1}{2}|M_4^{PT}|^2$$

which agrees with the result in equation (12).

For n=5, gg $\rightarrow$ ggg, the final state gluon labelled (1,2,3) and then, the subsets between two gluons interaction in the final states are $\{1,2\}$; $\{1,3\}$ and $\{2,3\}$. The number of the subsets on 3, and we have 3 final jets. Then, $P_5 = \frac{3}{3}$ which is one.

In this case, one writes equation (14) as $|M_5|^2 = P_5 \left|M_5^{PT}\right|^2 = \left|M_5^{PT}\right|^2$ which agrees with the known exact result.

For n=6, gg $\rightarrow$ gggg, denoting the final state gluon by (1,2,3,4) and then, the subsets between two gluons interactions in final states are $\{1,2\}$; $\{1,3\}$; $\{1,4\}$; $\{2,3\}$; $\{2,4\}$ and $\{3,4\}$. The number of the subsets between two final gluons on 6 and we have four final jets. Then $P_6 = \frac{6}{4}$ which is 1.5.

For n=7, gg $\rightarrow$ ggggg, the final state gluons are labelled (1,2,3,4,5) then, the subsets between two gluon interactions pains in the final states are $\{1,2\}$; $\{1,3\}$; $\{1,4\}$; $\{1,5\}$, $\{2,3\}$; $\{2,4\}$; $\{2,5\}$; $\{3,4\}$; $\{3,5\}$; $\{4,5\}$. Then, the number of the subsets is 10 and we have five final jet. In this case, the proportional factor can be written as $P_7 = \frac{10}{5}$ which is 2. Then, we can obtain a general form for our factor that was written as $P_n = \frac{1}{2}(n-3)$ and we rewrite the equation which was given in (14) as the following simple form:

$$|M_n|^2 \cong \frac{1}{2}(n-3)\left|M_n^{PT}\right|^2 \tag{15}$$

As we noted, the proportional factor is increasing by increasing the number of the final state gluons. The combinational factor of equation (11) is given very large values for n = 6, 7, 8, but the average values of the collinearity factor of equations (9) and (10) gives slow variation by increasing the number of the final states. But the proportional factor of equation (14) gives a reasonable variation.

Results and Discussion

It is easy to write computer codes to generate the approximate expression, which was represented in equation (15). The dominant time taken in these programs is that needed to sum over the (n-1)!/2 in the Parke and Taylor formula.

In Table 1, we compared the proportional factor, Pn, with the combinatoric factor, Cn, of ref. [3] and the collinearity factor, Fi, of ref. [1].

Table 1. Shows the proportional factor compared with combinatoric factor, Cn, and the Collinearity factor, Fi

Process	Pn	Cn	<nFi>
$gg \rightarrow gggg$	1.5	1.67	1.31
$gg \rightarrow ggggg$	2	2.67	1.56
$gg \rightarrow gggggg$	2.5	4.25	1.86
$gg \rightarrow ggggggg$	3	6.83	2.16
$gg \rightarrow gggggggg$	3.5	11.13	2.9

For comparison of these approximation with exact matrix elements squared 2 and 3, we did the Monte Carlo program with the same events in each case. The calculations are done at the Fermilab Tevatron 1800 Gev c.m. Energy, using the parton density functions with L = 0.2 Gev and as was used in the first order with Nj = 5.

We applied the following cuts on the outgoing partons, to get more or less realistic situations

$$P_i^T > 25\text{GeV}$$
$$|\eta_i| < 3.5$$

Where, P_i^T denotes the transverse momentum and h the rapidity of the outgoing gluon, and for each pair of the final jet $\cos\phi_{ij} < \cos\left[\frac{200}{n-2}\right]$.

Table 2. The total cross-section rate (in nb) for different aproximations, compared with exact cross-section rate for $gg \rightarrow gggg$

Method	$gg \rightarrow gggg$
Exact	15.8 ±. 20
Maxwell	15.0 . ± 20
K. Striling	19.1 . ± 30
Makhshoush	16.0 . ± 20

Table 3. The total cross-section rate (in nb) for different approximations, compared with exact cross-section rate for $gg \rightarrow ggggg$

Method	$gg \rightarrow ggggg$
Exact	.91 → .03
Maxwell	.81 → .03
K. Stirling	K. 1.33 → .04
Makhshoush	.90 → .03

The total cross-section rates for gg → gggg and gg → ggggg scattering are listed in Tables 2 and 3. The approximation of ref. [3] indicates that the Parke and Taylor amplitudes are greater than the other amplitudes. That only leading order in the number of colors was used apparently makes it only smaller. Both Makhshoush and Maxwell are reliable approximations. It is interesting to note that for gg → ggggg the approximation which was suggested by Makhshoush is more accurate than for gg → gggg. But Maxwell approximation is doing worse compared to gg → gggg case. This means that for the reduction method, the error increases from 5% for gg → gggg to about 11% for gg → gggg, making it dangerous to say something about the correctness for gg → 7g, gg →gg etc. For Makhshoush method the error is 3% for n=6 and 1% for n=7. This means that the approximation that was represented in equation (15) might work well for n=8,9,10 etc.

References

[1] Maxwell, C. "Approximation n-jet Production." *Phys. Lett.*, B 192 (1987), 190-194.\
[2] Parke, S. and Taylor, T. "Amplitude for n-gluon Scattering" *Phys. Rev.* Lett., 56 (1986), 2459-2461.
[3] Kunszt, Z. and Stirling, W. "Proceeding of the Workshop on Phys. at Future Accelerator." La Thuile and Geneva, J.H. Mulrey (Ed.), 2 (1987), 548-553.
[4] Kunszt, Z. "Combined Use of the Calkul Method and N=1 Supersymmetry to Calculate QCD Six-parton Processed." *Nucl. Phys.*, B271(1986), 333-348.
[5] Kleiss,R. and Kujf, H. "Multigluon Cross-Section and 5-jet Production at Hadron Colliders." *Nucl. Phys.*, B 312 (1989), 616-626.

تقريب التشتت لـ – ن جلوُون

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ملخص البحث. في هذا البحث أعطينا صيغة تقريبية لتشتت ن – جلووُن. هذا التعريف يرتكز على صيغة بارك – تايلور (مع إهمال حدود التصحيح) ومعامل التناسب.

يعطي التقريب المقترح في هذا البحث توافق ممتاز في حالة تشتت ستة جلووُنات وسبعة جلووُنات إذا ماقورن بالنتائج التامة المصنفة في المراجع.

باستخدام التقريب الذي توصلنا إليه في هذا البحث وجدنا أن نسبة الخطأ تنخفض إلى حوالي ١٪ مقارنة مع النتائج التامة.